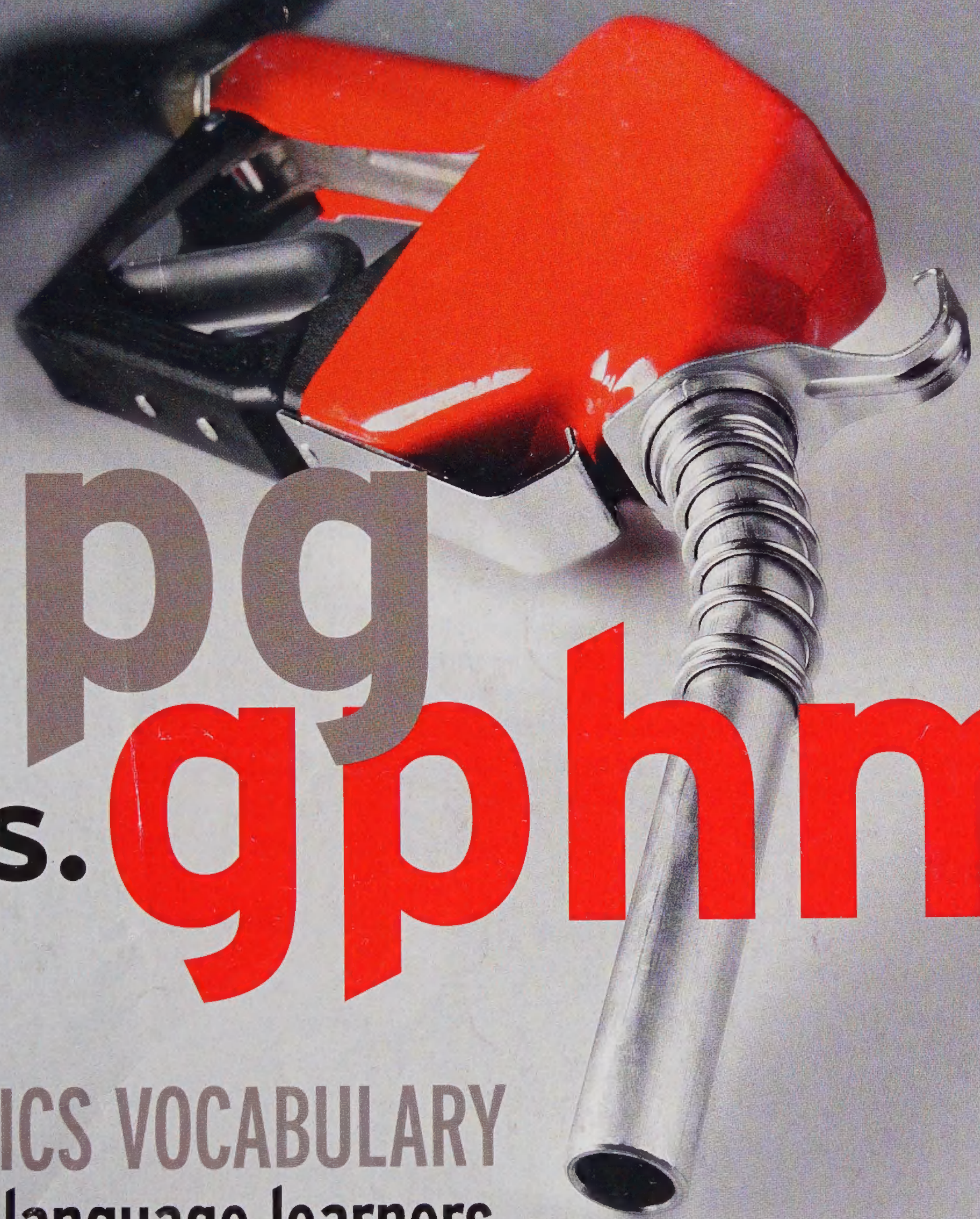
NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS | WWW.NCTM.ORG

MATHEMATICS teacher

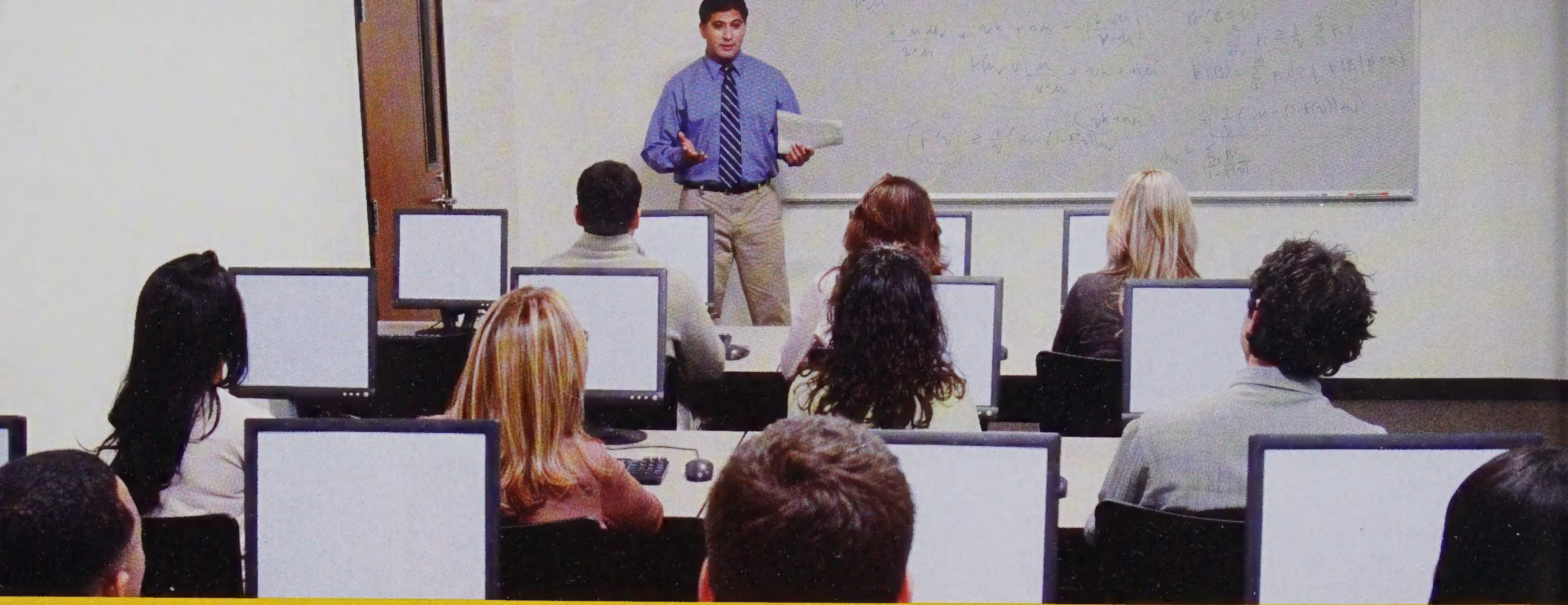
AUGUST 2013

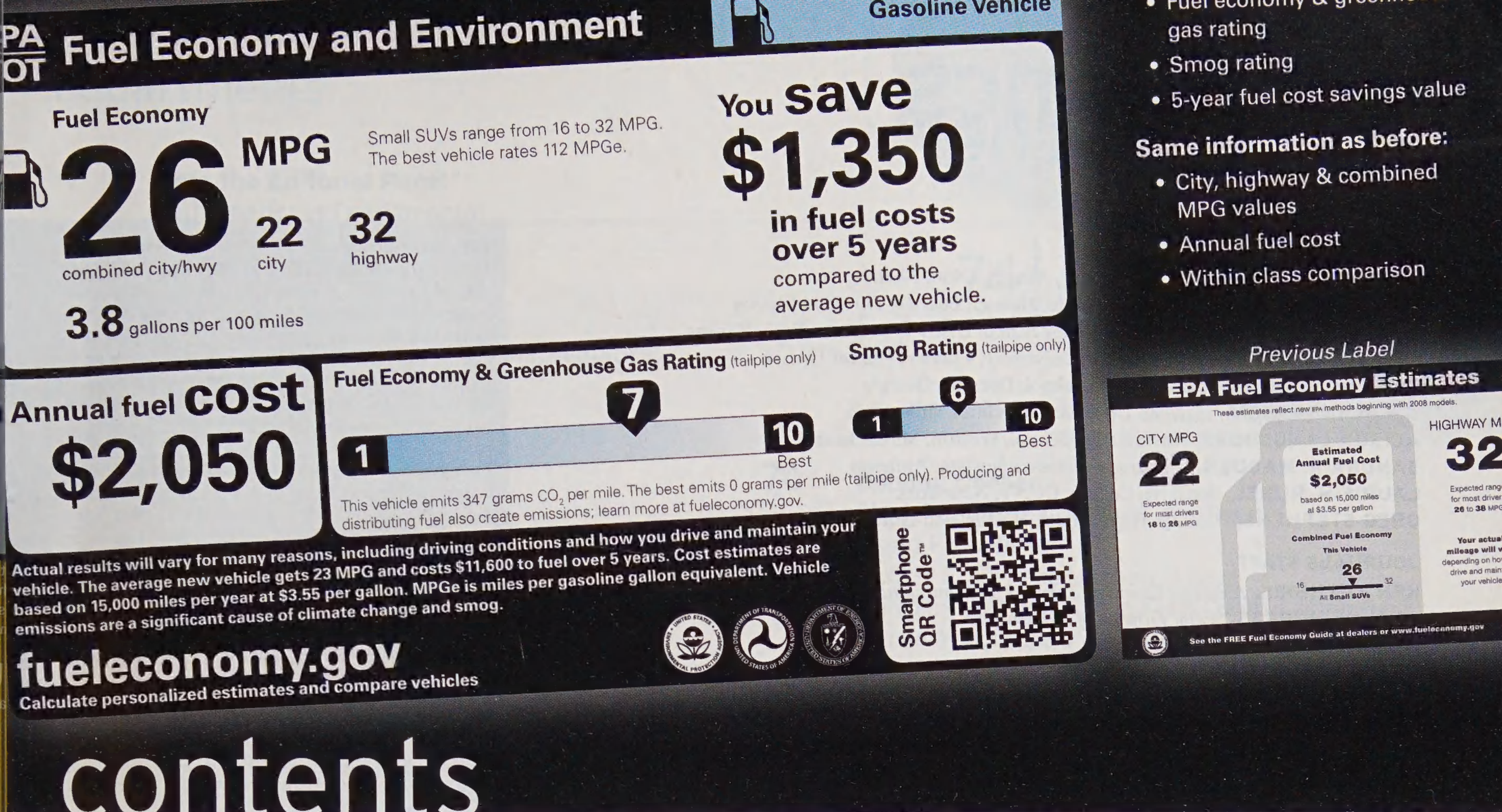


mpg vs. gphm

MATHEMATICS VOCABULARY
and english language learners

APPLYING MATRICES
digital image processing





contents

Volume 107, Number 1, August 2013

features

on the cover

Which saves more fuel—converting a car that gets 10 mpg to one that gets 11 mpg or converting a car that gets 38 mpg to one that gets 50 mpg? Before answering, think about the Cash for Clunkers program and then turn to page 20. There Kevin G. Bartkovich provides not only the answer but also a rich lesson chock-full of mathematics and tied to the environment and other real-world concerns.

COVER PHOTO: FRANCIS BLACK/ISTOCKPHOTO

NCTM NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

MATHEMATICS teacher

To contact a journal staff person, e-mail **mt@nctm.org**.
For NCTM advertising sales, contact **The Townsend Group,** Kim Kelemen, national sales manager; Kkelemen@townsend-group.com; (301) 215-6710, ex. 103

Mission Statement: The National Council of Teachers of Mathematics is the public voice of mathematics education, supporting teachers to ensure equitable mathematics learning of the highest quality for all students through vision, leadership, professional development, and research.

Mathematics Teacher, an official NCTM journal, is devoted to improving mathematics instruction for grades 8–14 and supporting teacher education programs. It provides a forum for sharing activities and pedagogical strategies, deepening understanding of mathematical ideas, and linking mathematics education research to practice. *Mathematics Teacher* solicits submissions from high school mathematics teachers, university mathematicians, and mathematics educators and strongly encourages manuscripts in which ideas relate to classroom practice. Manuscripts previously published in other journals or books are not acceptable. NCTM publications present a variety of viewpoints. The views expressed or implied in this journal, unless otherwise noted, should not be interpreted as official NCTM positions. The appearance of advertising in NCTM's publications and on its websites in no way implies endorsement of approval by NCTM of any advertising claims or of the advertiser, its product, or its services. NCTM disclaims any liability whatsoever in connection with advertising appearing in NCTM's publications and on its websites.

All correspondence should be addressed to the *Mathematics Teacher*, 1906 Association Drive, Reston, VA 20191-1502. Manuscripts should be prepared according to the *Chicago Manual of Style* and the United States Metric Association's *Guide to the Use of the Metric System*. No author identification should appear on the manuscript; the journal uses a blind-review process. To send submissions, access **mt.msubmit.net**. Send letters to the editor to **mt@nctm.org**.

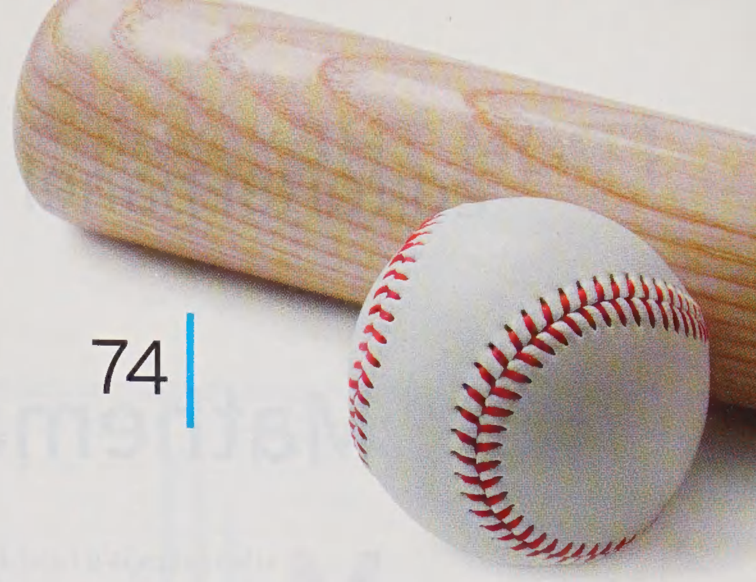
For information on **article photocopies or back issues**, contact the Customer Service department in the Headquarters Office.

A cumulative index appears on the NCTM website at **www.nctm.org/mt/mt-indexes.htm**. The *Mathematics Teacher* is indexed in *Biography Index, Contents Pages in Education, Current Index to Journals in Education, Education Index, Exceptional Child Education Resources, Literature Analysis of Microcomputer Publications, Mathematical Reviews, Media Review Digest,* and *Zentralblat für Didaktik der Mathermatik.*

Information is available from the Headquarters Office or online at www.nctm.org/membership regarding the three **other official journals**, *Teaching Children Mathematics, Mathematics Teaching in the Middle School*, and the *Journal for Research in Mathematics Education*. Dues support the development, coordination, and delivery of NCTM's services. Dues for individual membership are \$81 (U.S.), which includes \$34 for each *Mathematics Teacher* subscription. Each additional school journal (*Mathematics Teaching in the Middle School* and *Teaching Children Mathematics*) subscription is \$34. Each additional subscription to the *Journal for Research in Mathematics Education* is \$61. Foreign subscribers add \$18 (U.S.) postage for the first journal and \$4 (U.S.) postage for each additional journal. Special rates for students, institutions, bulk subscribers, and emeritus members are available from the Headquarters Office.

The *Mathematics Teacher* (ISSN 0025-5769) (IPM 1213431) is published monthly except June and July, with a combined December/January issue, by the National Council of Teachers of Mathematics at 1906 Association Drive, Reston, VA 20191-1502. Periodicals postage paid at Herndon, Virginia, and at additional mailing addresses. POSTMASTER: Send address changes to *Mathematics Teacher*, 1906 Association Drive, Reston, VA 20191-1502. Telephone: (703) 620-9840; orders: (800) 235-7566; fax: (703) 476-2970; e-mail: nctm@nctm.org.

departments



74

calls for manuscripts

more4U

Look online for these additional items:

- "Modeling Fuel Efficiency: MPG or GPHM?" (p. 20): Calculating the CAFE Standard
- "Learning with Calculator Games" (p. 35): Codes for calculator games plus activity sheets
- Reader Reflections (p. 5): Excel file for coin tossing
- Activities for Students (p. 62): Customizable activity sheets

Download one of the free apps for your smartphone. Then scan this tag to access **www.nctm.org/mt.**

in the next issue

Coming in September 2013:

- "The Geoboard Triangle Quest," by Kasi C. Allen
- "Exploring Function Transformations Using the Common Core," by Becky Hall and Rich Giacin
- "Adding a New Dimension to Algebra," by Sheldon P. Gordon
- "I Need More Information!" by Christine P. Trinter and Joe Garofalo

Mathematics Is Fundamental

Mathematics is a basic human activity. Next to learning one's mother tongue, learning to count, measure, and recognize and use shapes is fundamentally important. Many of the words most frequently used in any natural language are related to things mathematical:

- Number (*a*, *an*, *one*, *two*, *no*, *none*, *all*, *some*, *many*, *few*, *first*, *each*, *both*)
- Time (*day*, *year*, *before*, *after*)
- Measurement (*large*, *small*, *big*, *long*, *short*, *tall*, *more*, *less*)
- Logic (*and*, *or*, *but*, *not*, *if*, *then*)
- Uniqueness (*the*, *only*)

Moreover, mathematics provides answers to many questions: How many? How much? How often? How far? When? Where? As mathematics teachers, we have chosen as our life's work inspiring the next generation to understand and embrace the importance of mathematics in their lives and helping learners answer life's mathematical questions.

Like mathematics itself, teaching is a fundamental human activity. Without teaching, the world as we know it would not exist. Teachers help young people develop knowledge and skills to function in our complex society. Mathematics teachers in particular play a central role in the formation of citizens, and they develop quantitative, visual, and logical thinking among young people.

The Editorial Panel of *Mathematics Teacher:* (back row) David Custer, Greg Stephens, Greg Foley, Dane Camp; (middle row) Steve Ingrassia, Alison Langsdorf, Sandra Madden, Albert Goetz; (front row) Laurie Rubel, Dan Canada, Elizabeth Appelbaum.

NCTM's reasoning and sense-making focus and the Common Core's Standards for Mathematical Practice outline frameworks to engage students in rich tasks within a thinking curriculum. These standards describe a secondary school curriculum that is grounded in number, quantity, algebra, functions, and geometry but that extends to modeling, discrete mathematics, probability, and statistics. For more than a century, *Mathematics Teacher* has provided a teacher-to-teacher forum that offers a level of specificity not found in such policy documents. The *MT* Editorial Panel hopes that *Mathematics Teacher* will serve as a helpful companion in your daily work of engaging and inspiring students.

The *MT* departments—Activities for Students, The Back Page, Calendar, Connecting Research to Teaching, Delving Deeper, For Your Information, Mathematical Lens, Media Clips, Reader Reflections, Sound Off!, and Technology Tips—are designed to evolve to serve the diverse needs and interests of readers. The Problem Solvers Department has morphed into a Problem of the Month in the Calendar. Readers are encouraged to have their students solve these problems and submit solutions to the Problem of the Month editors for possible publication in the journal. In another change, the Calendar will revert to representing the publication month; in other words, the Calendar will run from August through May.

The 2013 *MT* Focus Issue has as its theme Beginning Algebra: Teaching Key Concepts. This timeless topic will be spotlighted in the November issue, and numerous other feature articles throughout the volume will address the teaching of algebra. Other issues will feature articles on infinity, Angry Birds, the Stable Marriage problem, and lots of others.

As the new school year begins, the Editorial Panel offers you best wishes for a productive and rewarding year as you engage your students in new challenges in mathematics, statistics, and modeling. And we hope that you have lots of fun along the way!

EXCEL AND VISUAL BASIC

In "Investigating the Law of Large Numbers with Visual Basic" (Technology Tips, *MT* Sept. 2012, vol. 106, no. 2, pp. 144–49), Nicole R. Juersivich demonstrates how Excel® can be used to investigate the law of large numbers. Excel does most of the work. Only two short macros are needed. If you are willing to write a longer macro using the **Cells** command, you can achieve the same result using only one worksheet.

Figure 1 (Janovsky) contains a macro for coin tossing that uses the **Cells** command. I like the **Cells** command because it allows me to capture and move data easily. (An Excel file can be found at **www.nctm.org/mt**.)

Andrew Janovsky
janovsky1@optonline.net
John F. Kennedy High School (retired)
The Bronx, NY, Oct. 22, 2012

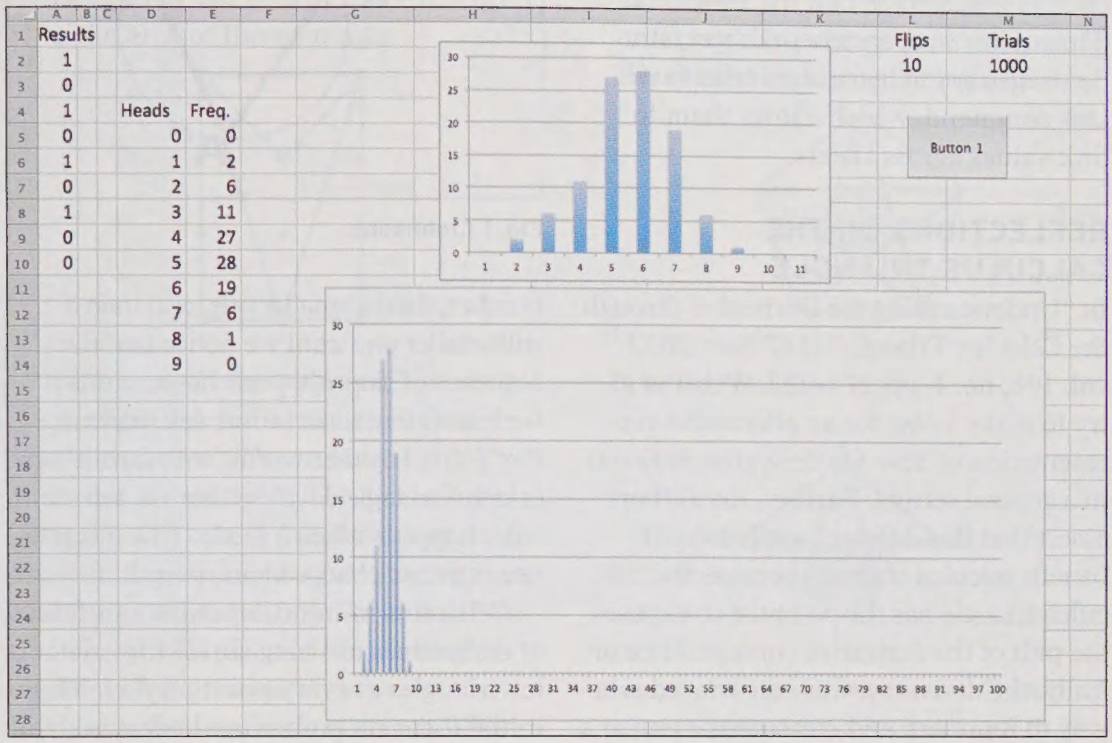


Fig. 1 (Janovsky)

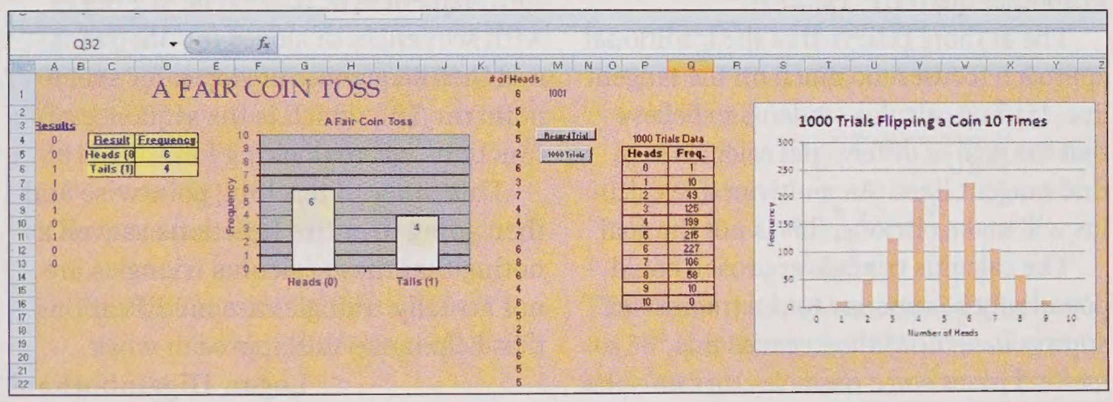


Fig. 1 (Juersivich)

Juersivich replies: Yes, the **Cells** command is a way to move data easily. However, even if you use the macro code in the article, you can still fit everything on one worksheet (see **fig. 1 (Juersivich)** for a screenshot of one of the two Excel files available at **www.nctm.org/mt**). I purposefully chose not to put everything on one worksheet when doing this lesson.

The worksheet becomes visually busy if it includes too much information. In my experience, students have trouble focusing on so many things that are changing at once. Further, if the worksheet is all on one page, the information on the sheet extends outside the computer screen (i.e., outside students' perception field) unless students zoom out—but doing so causes the values and graphs to become almost unreadable.

One misconception that this applet corrects is the idea of a trial versus a sequence of flips. I want students to be able to see across multiple representations—numerically, tabularly, and graphically—and understand how the outcomes of each sequence of flips contributes to a single trial. Further, I want students to become comfortable with how the technology displays one trial of a sequence of 10 flips by using the results column, a frequency table for one trial, and a histogram for one trial. I then want students to realize that one benefit of technology is the ability to record the data for us.

When I first came across this lesson, students had to record all the data manually, prompting me to look into macros in the first place. When students are able to record one trial of the number of heads with the **Record Trial** button, they are able to see how the computer is transferring the number of heads from the trial into column L.

This step may seem simple, but I have found that when the computer does too many things at once, students tend to think that the technology is magical, a

We appreciate the interest and value the views of those who write. Readers commenting on articles are encouraged to send copies of their correspondence to the authors. For publication: All letters for publication are acknowledged. Letters to be considered for publication should be in MS Word document format and sent to **mt@nctm.org**. Letters should not exceed 250 words and are subject to abridgment. At the end of the letter include your name and affiliation, if any, including e-mail address, per the style of the section.

perception that tends to increase their reliance on technology. By having a record 1 trial button, students can see that Excel records the number of heads data, and they realize the power of being able to run 1000 trials at once with a click of a button. When the 1000 trial data are on a separate worksheet, students also get to learn about the **Paste Link** command, which allows them to link values across sheets.

REFLECTIONS ON THE CALCULUS TRIANGLE

In "Understanding the Derivative through the Calculus Triangle" (*MT* Nov. 2012, vol. 106, no. 4, pp. 274–78), Weber et al. try to make a case for an alternative representation of how the derivative is found in a general setting. Further, the authors assert that this different approach will benefit calculus students because the calculus triangle has the potential to expose the pith of the derivative concept: "Historically, the derivative was constructed as a way to represent and measure the rate at which one quantity changes with respect to another quantity" (p. 275).

The authors believe that the traditional approach focuses too much on the tangent line, leading calculus students to believe that the goal of differential calculus is to find tangent lines. As multivariable calculus will make obvious, this is not the goal.

The calculus triangle approach could potentially be superior to the traditional approach—*potentially* because this method poses some obstacles that must be overcome. Weber et al. do mention two: (1) students seeing the calculus triangle as an actual triangle and (2) students being unable to envision multiple calculus triangles at one time.

The first obstacle is not an obstacle at all. Teachers simply need to point out that the calculus triangle is not a real triangle.

The second obstacle can be avoided by introducing pointwise derivatives before functional derivatives. Note that I am using what was advantageous for the traditional approach (pointwise derivatives) without mentioning tangents. For example, if we wish to look at the derivative of $f(x)$ at the point x, we make a calculus triangle involving some positive real number—say, 3—and compute the slope at those points. Then we repeat the process with a smaller number, then a smaller one, and then a still smaller one until we notice that the sequence of slopes approaches a number we know; that must be our derivative at that point. In other words, we "shrink" our calculus triangle down so that we can see what happens when h is zero (the instantaneous rate of change at one point).

With this in mind, students can think of derivatives as using shrinking calculus triangles at every point on $f(x)$. This initial focus on pointwise derivatives will also make it easier for students to envision sequences of functions: We begin with sequences of slopes for one point and then sequences of slopes for every point on $f(x)$, which is the sequence of functions we are looking for.

This order of teaching pointwise and then going to entire functions as well as noting that these calculus triangles are not actually triangles are modifications that help make this approach work.

Logan Higginbotham
zagmanfan1@yahoo.com
Morehead State University
Morehead, KY, Dec. 3, 2012

HISTORY OF FERMAT'S POINT

In "Authentic Tasks in a Standards-Based World" (*MT* Dec. 2012/Jan. 2013, vol. 106, no. 5, pp. 346–53), Edwards, Harper, and Cox referred to a method for finding Fermat's point. The method shown was attributed to Evangelista Torricelli (1608–47), one of many mathematicians with whom Fermat corresponded. In fact, Fermat was the clearinghouse for much of European mathematics in the seventeenth century. He typically challenged his correspondents with problems such as Fermat's point and would often post solutions for all to critique.

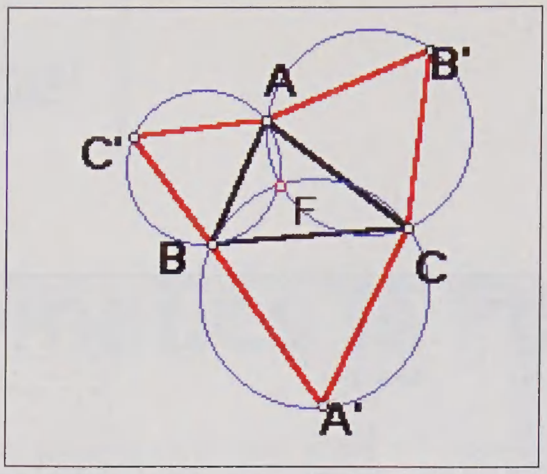


Fig. 1 (Johnson)

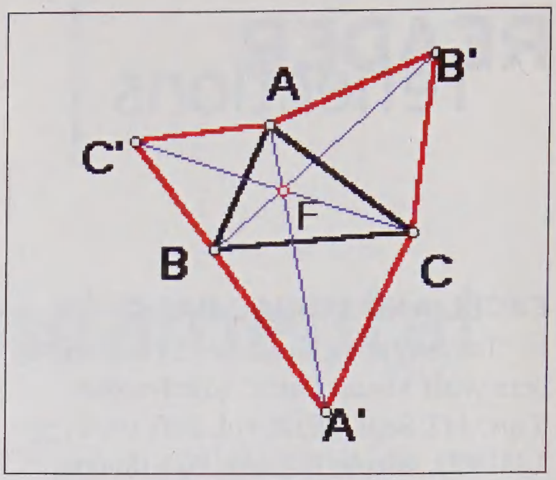


Fig. 2 (Johnson)

Torricelli did send a solution to Fermat, but it involved constructing circumscribed circles about the three equilateral triangles formed on each leg of the original triangle. The three circles intersect at Fermat's point (see **fig. 1 [Johnson]**). The method shown in the article is attributed to Bernard Frenicle de Bessey (1602–75) (see **fig. 2 [Johnson]**).

Art Johnson
johnsonart@aol.com
Boston University
Boston, Jan. 13, 2013

PROBLEM 21, MAY 2012 CALENDAR

Problem 21 of the May 2012 Calendar (see *MT* April 2012, vol. 105, no. 8, p. 601) states:

> Right triangle ABC has a right angle at C. M is the midpoint of $\overline{AB}$. If $BC = 7$ and $MC = 12.5$, find $\sin(\angle CMB)$.

This problem can be construed to mean that a right triangle is completely determined if we are given the length of one leg and the distance, call it q, from the vertex of the right angle to the midpoint of its hypotenuse. Essentially, the reason is that the midpoint is the center of the circumscribed circle, thereby making the length of the hypotenuse $2q$. This interpretation motivates a consideration of the situation where the triangle does not necessarily contain a right angle. (Throughout this discussion, we assume that we are dealing with a given set of values of the variables that leads to a viable, unique solution. I have not investigated necessary conditions for such.)

Suppose that the three values given are p, q, and r where M is a midpoint

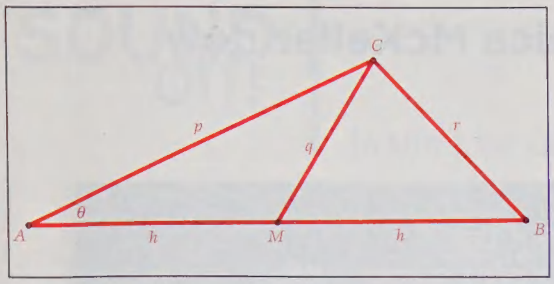


Fig. 1 (Siegel)

of side AB (see **fig. 1 [Siegel]**). Let us show that this is sufficient information to determine the triangle. To that end, the law of cosines, applied to triangles MAC and BAC, respectively, gives us $q^2 = p^2 + h^2 - 2ph\cos\theta$ and $r^2 = p^2 + (2h)^2 - 2(2h)p\cos\theta$. On eliminating θ, we obtain the relationship that can be used to determine $2h$, the length of the third side of the triangle: $r^2 + p^2 = 2(q^2 + h^2)$.

We might ask whether the triangle is determined if M is not a midpoint but rather x units from A. In that case, where s represents the length of side AB, our equations become $q^2 = p^2 + x^2 - 2px\cos\theta$ and $r^2 = p^2 + s^2 - 2sp\cos\theta$. Eliminating θ, we obtain $xs^2 + (q^2 - p^2 - x^2)s + x(p^2 - r^2) = 0$, which determines s.

Steven Siegel
siegel443@comcast.net
Niagara University (retired)
Feb. 13, 2012

PROBLEMS 1, 20, AND 21, DECEMBER 2012 CALENDAR

Problem 1

Problem 1 of the December 2012 Calendar (*MT* Nov. 2012, vol. 106, no. 4, p. 280) states:

> The coordinates of the endpoints of a diameter of a circle are (–2, 7) and (6, 11). Write an equation of the circle.

Any point $P\,(x, y)$ on the semicircle, taken together with $A\,(-2, 7)$ and $B\,(6, 11)$, will form a right triangle. Using the Pythagorean theorem, we get

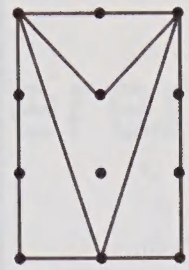
$$\begin{aligned} PA^2 + PB^2 = AB^2 \to \\ (x+2)^2 + (y-7)^2 \\ + (x-6)^2 + (y-11)^2 = \\ (6+2)^2 + (11-7)^2. \end{aligned}$$

This simplifies to $x^2 + y^2 - 4x - 18y + 65 = 0$, which is equivalent to $(x - 2)^2 + (y - 9)^2 = 20$.

Problem 20

Problem 20 of the December 2012 Calendar (*MT* Nov. 2012, vol. 106, no. 4, p. 281) states:

> Given the 12 lattice points, determine the ratio of the area of the arrowhead to the area of the rectangle.



If the arrowhead is shaded, the area of the unshaded region consists of three triangles, two of which are congruent. The unshaded area = $2(1/2)(a)(3a) + (1/2) \cdot (2a)(a) = 4a^2$, where a is the distance between any two consecutive vertical or horizontal lattice points. The area of the arrowhead = the area of the rectangle – the unshaded area: $6a^2 - 4a^2 = 2a^2$.

Hence, the required ratio is $2a^2/6a^2 = 1/3$.

Problem 21

Problem 21 of the December 2012 Calendar (*MT* Nov. 2012, vol. 106, no. 4, p. 281) states:

> Compute the sum:
>
> $$\sin(\pi/7) + \sin(4\pi/7) + \sin(7\pi/7) + \sin(10\pi/7) + \sin(13\pi/7)$$

Using the formula, we find that

$$\sin A + \sin B = 2\sin\left(\tfrac{A+B}{2}\right)\cos\left(\tfrac{A-B}{2}\right).$$

Grouping terms as follows, we obtain the following:

$$\begin{aligned} &\sin(13\pi/7) + \sin(\pi/7) \\ &\quad = 2\sin(\pi)\cos(6\pi/7) = 0 \end{aligned}$$

$$\begin{aligned} &\sin(10\pi/7) + \sin(4\pi/7) \\ &\quad = 2\sin(\pi)\cos(3\pi/7) = 0 \end{aligned}$$

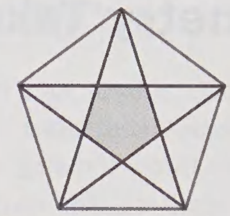
Since the middle term, $\sin(7\pi/7)$, equals $\sin\pi = 0$, the sum is 0.

Jeganathan Sriskandarajah
jsriskandara@madisoncollege.edu
Madison College
Madison, WI, Nov. 13, 2012

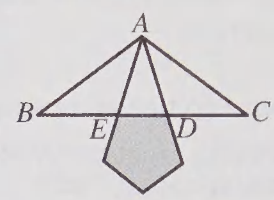
PROBLEM 25, DECEMBER 2012 CALENDAR

Problem 25 of the December 2012 Calendar (see *MT* Nov. 2012, vol. 106, no. 4, p. 281) states:

> Determine the ratio of the area of the outer regular pentagon to the area of the shaded inner regular pentagon.



The first solution given—



—has an intriguing byproduct. Dropping a perpendicular from A to $\overline{BC}$ at F to form right triangle AFC gives

$$\begin{aligned} \cos 36^\circ &= \frac{CF}{CA} = \frac{b+\frac{1}{2}}{b+1} \\ &= \frac{\phi+\frac{1}{2}}{\phi+1} = \frac{2+\sqrt{5}}{3+\sqrt{5}} \\ &= \frac{1+\sqrt{5}}{4} = \frac{\phi}{2}. \end{aligned}$$

This computation shows that the value of $\cos 36^\circ$ is not only exactly one-half φ but also that $\cos 36^\circ$ is not transcendental but algebraic.

David Bernklau
davidbee2009@gmail.com
Long Island University
Brooklyn, NY, Dec. 28, 2012

SOUND off!

Jason Lee O'Roark

The Myth of Differentiation in Mathematics: Providing Maximum Growth

After teaching high school mathematics in Maryland for three years, I began teaching sixth-grade mathematics in one of the best school districts in Pennsylvania (according to state test scores) and have been teaching there for the past six years. My high school teaching background led me to differentiate differently from my colleagues. I share my observations of the result of the differences in methodology and my conclusions from those observations, and I offer a plan to implement changes in the way that mathematics is taught.

TYPICAL MIDDLE SCHOOL MATHEMATICS TEACHING PRACTICES

The school district in which I teach uses heterogeneous grouping in the middle schools for all students except for the top few percent of mathematics students, who are placed in accelerated classes. Differentiation within the heterogeneous classes is based on pretests before each chapter that determine whether a student already knows the material. Common extensions include application projects, more word problems, and activity sheets with larger numbers or a slightly advanced twist on the concept. Providing these types of extensions ensures that the base knowledge of the students in the class remains similar, allowing the district to continue to apply its standard curriculum while appearing to meet all students' needs through differentiation.

However, is the mathematical growth of advanced students as significant as that of average students? Are below-average students really getting the instruction that is most appropriate for them? The extensions we provide for advanced students may help keep them busy and may make them think, but the real effect of such extensions is to slow these students down until the rest of the class catches up. This kind of instruction provides little motivation to advanced students. Meanwhile, lower-ability students find mathematics frustrating as they attempt to learn concepts that build on ideas that they never fully grasped.

Sound Off! is *Mathematics Teacher*'s op-ed page; as such, the opinions expressed reflect those of the author and not necessarily those of the *MT* Editorial Panel or of the National Council of Teachers of Mathematics. Readers are encouraged to respond to this Sound Off! by submitting letters to the Reader Reflections section of the journal as well as to submit essays for consideration as Sound Off!s. Please visit http://www.nctm.org/publications/content.aspx?id=10440#soundoff for information.

PROVIDING STUDENTS WITH TRUE GROWTH OPPORTUNITIES

My previous experience with teaching higher-level mathematics has enabled me to provide what I think are true growth opportunities within the sixth-grade mathematics classroom. Rather than delay advanced students, I give them access to higher-level material. My class begins the school year in the same way as all the other mathematics classes in the school, but by the end of the school year I am teaching different mathematics topics simultaneously, a result of the students' divergent paths.

I am not trying to promote my approach as a superior method of education. However, when I differentiate instruction, students have shown that they can do amazing things. My sixth-grade students begin the year not knowing how to solve a one-step equation. By the end of the year, some of them can solve systems of equations with four variables and derive the quadratic formula by completing the square—skills that are two to four years beyond their current curriculum. And they love it.

When motivated students are allowed to progress at their own pace, they enjoy mathematics. Many are capable of so much more than what a standard curriculum provides. How many students could take calculus in the first year of high school if they were not kept at the same pace as their peers? At the other end of the spectrum, I have taught quadratic equations to students who struggle with basic mathematics because they

need to pass a state test of first-year algebra. Shouldn't they be given the opportunity to grow in areas of mathematics that are more appropriate for them?

REFLECTIONS ON THESE CONTRASTING METHODS

Which nation will be the more prosperous—a nation that ensures that all its citizens meet a minimum standard of education or a nation that strives for the maximum educational growth that each student can achieve? The United States has been the former, trying to ensure that no child is "left behind."

Imagine, instead, an education system that encourages students to experience as much true mathematical growth as possible, regardless of age. Imagine an education system that delivers instruction at exactly the appropriate level and in the appropriate area for each student, rewards students for their brilliance and hard work by allowing them to progress to higher-level mathematics, and works with lower-ability students to ensure mastery of key concepts without feeling the pressure of time constraints.

Students of the same age would be in very different places mathematically. No students would have gaps in their mathematical understanding. All would be mathematically prepared to provide the highest level of contribution to society that they are capable of. Achieving this kind of system requires two steps: sufficient motivation for education institutions to change and a plan to make such change possible.

MOTIVATION FOR CHANGE: ASSESSMENTS

No Child Left Behind is succeeding in ensuring that all students are being educated. However, the assessments that this program has spawned are advancing educational policies that are detrimental to the nation.

Current assessments ensure that all students have met a minimum standard for their grade level, leading schools to focus their energies on the least able students. We give the most educational attention to the students least likely to use it. Although this approach may seem humanitarian, it fails to produce what our country needs.

Instead, all state assessments should be growth-based, giving schools credit for how much students have grown—that is, how much material they have mastered. This kind of assessment would change the schools' focus: Producing the most growth possible for all students would be the goal. Striving to reach that goal could revitalize our nation.

State tests should be computerized tests that determine the development level of students in all areas of mathematics. The assessment would start in one branch of mathematics at a level slightly below the expected level, as set by the school district, and progress upward until a student could no longer correctly answer questions. At that point, students would move on to questions at the expected level in the next branch of mathematics. Schools would be evaluated on the amount of growth exhibited by students compared with an expected growth norm that may take into account predetermined developmental or socioeconomic factors.

A PLAN: MAXIMIZING THE GROWTH OF EVERY STUDENT

Teaching effectively to prepare students for a growth-based assessment would be difficult. If the goal is to allow students to "be all they can be" and provide for each one's maximum possible growth, then every student will learn different amounts of material every year. As a result, students of the same age will be years ahead of or behind other students their age. The traditional classroom no longer seems practical. Deciding how best to approach this challenge will require much thought and discussion.

My main point is that educating for maximum growth for every student is superior to ensuring that every student meets a minimum standard. Here is one plan that seems logical.

I believe that a *really good* computer program could do a better job of meeting students' needs than I can while trying to teach a whole class at once. Many programs currently available—such as Cognitive Tutor, Study Island, Apex Math, SmartMath, IXL, and ALEKS—are moving in a positive direction, but none of them is there yet. However, I am confident that, given sufficient resources, a program capable of transforming mathematics education can be produced, perhaps in cooperation with one of the programs mentioned.

A quality computer program would include these properties:

- It gets to know the student as the student is learning mathematics—his or her likes, dislikes, struggles, topics to review, strengths, and so on.
- It engages and motivates, providing the student with levels.
- It incorporates games and allows the student to earn rewards. I envision a sim-civilization game in which the civilization develops with students' mathematical growth—from addition (allowing commerce with their classmates) to quadratics and trigonometry (helping them aim their catapults).
- It quickly diagnoses problems and moves within the web of the vast curriculum to aid the student.
- It ensures continued mastery through spiraled checks.
- It allows students to work within the program at any time, from anywhere. If students want to put in extra work to progress mathematically, they can. This flexibility will allow lower-ability students to put in more work to find success and allow motivated students to excel.
- It provides easy-to-use reports to aid in monitoring progress.
- It gives official assessments in secure locations to ensure that the student is the person making the progress.
- It has hands-on application opportunities woven throughout its framework.

Creating such an extensive program would require the best programmers who design popular games to work alongside expert educators. The amount of time and money needed would seem to be a huge obstacle. However, given the financial rewards for producing a program so good that nearly every student in the nation would want to use it, such a program would easily pay for itself. In any case, something truly transformative could be created, and doing so would be worth the time, effort, and money.

In this more individualized approach, teachers would still have important roles, but they would be very different from their roles today. Teachers would serve as another resource to students as they work; provide hands-on application and collaboration opportunities; monitor progress; provide coaching, encouragement, and support; and create a supportive learning community.

In some ways, teachers would need to be more qualified to teach in classrooms that use such a program than to teach in a traditional classroom. To be effective, teachers would need to be able to help students with several different years' worth of material without having the chance to review it, as they would before teaching a lesson. Teachers would also guide students in applying what they have learned.

EMPHASIZING THE INDIVIDUAL

Differentiation, individualized education, self-directed learning—we have only just begun implementing the concepts behind all these buzz words. It is time to take these concepts to the next level, to unlock all the potential of our students through truly customized learning. It is time to meet each student where he or she is and help all students grow. It is time to stop limiting students' growth to keep them on pace with their peers. It is time to stop forcing students to move on before they are ready.

We now have a tool that, if used properly, will allow education institutions to provide this kind of instruction. American students can rise to and surpass the levels of their international counterparts. We need to give them the right tools, support them, and then let them go.

JASON LEE O'ROARK, joroark@uscsd.k12.pa.us, teaches sixth-grade mathematics for Upper St. Clair School District in Pennsylvania. With a team of teachers, he also conducts the 5th-6th grade Calcu-Solve competition at Duquesne University in Pittsburgh.

Ron Lancaster

Lumbering Along

THE QUESTION: "Why are lumber measurements all incorrect?" asks G. Nicholson of Toronto. "For example, a piece of wood the hardware store bills as a 2-by-4 is actually more like 1.5 by 3.5 inches."

THE ANSWER: "The measurements are for unplaned lumber as it comes through the sawing process," writes Robert Lindsay of Peace River, Alta. "If you can find unplaned lumber, a 2-by-4 will be 2 inches by 4 inches." Planing the lumber smoothes its surfaces by removing a portion of the wood from each face of the board, he says, thus reducing the finished dimensions.

BARCNI/ISTOCKPHOTO

Source: *Collected Wisdom*, by Philip Jackman, (*Toronto*) *Globe and Mail*, May 29, 2010

1. (*a*) Compare the volume of a 1-ft.-long, unplaned 2 in. × 4 in. piece of wood with a planed piece of the same length, assuming that the planed piece has the dimensions stated in the question above. How much wood was removed during the sawing process?

(*b*) Let the height and width (in in.) of a 1-ft.-long piece of planed wood be represented by $(2 - x)$ and $(4 - x)$, respectively. Let y represent the volume of this planed piece of wood divided by the volume of a 2×4 of the same length. Without using an algebraic equation or technology, sketch a rough graph of y versus x.

(*c*) Determine an equation for y in terms of x. Use this equation to graph y versus x and compare it with the graph in part (*b*).

(*d*) Use the equation from part (*c*) to find the value of x for which half the volume of the wood is lost during the sawing process.

(*e*) Let the height and width (in in.) of a 1-ft.-long piece of planed wood be represented by $(2 - x)$ and $(4 - x)$, respectively. Let y represent the surface area of this planed piece of wood divided by the surface area of a 2×4 of the same length. Determine an equation for y in terms of x. Use this equation to graph y versus x.

(*f*) Use the equation from part 1(*e*) to find the value of x for which half the surface area of the wood is lost during the sawing process.

Media Clips appears in every issue of *Mathematics Teacher*, offering readers contemporary, authentic applications of quantitative reasoning based on print or electronic media. All submissions should be sent to the editors. For information on the department and guidelines for submitting a clip, visit http://www.nctm.org/publications/content.aspx?id=10440#media.

*Edited by **Louis Lim**, louis.lim1@gmail.com*
Thornhill Secondary School, Thornhill, ON, Canada

***Lionel Garrison**, garrison@horacemann.org*
Horace Mann School, Bronx, NY

Plans: Stairs and Handrails
Handrails must be graspable. In other words, the circular diameter of the handrail must not be greater than 2 in. and a minimum of 1 1/4 in. This is a round handrail. You can use another shape other than round, but the perimeter dimensions must be at least 4 in. and not more than 6 1/4 in. Example: A 2 × 4 piece of wood will not make the requirements for a handrail because its perimeter dimension is greater than the 6 1/4 in. maximum.

Source: "Constructing a Deck in Old Bridge Township, New Jersey," http://www.oldbridge.com/content/137/163/245.aspx

2. (*a*) Verify that a 2 × 4 piece of wood does not meet the requirements of a handrail in Old Bridge Township.

   (*b*) Let the height × width (in in.) of a 1-ft.-long piece of planed wood be $(2 - x) \times (4 - x)$. Find all values of x for which this piece of wood would meet the requirements.

Prices at Beardsley's Sawmill, Greene, New York		
Length (ft.)	**Unplaned 2 in. × 4 in.**	**Planed 1.5 in. × 3.5 in.**
8	$2.51	$3.04
10	$3.33	$4.00
12	$4.16	$4.96
14	$5.04	$5.97
16	$6.08	$7.15

Source: http://www.sawbiz.com/products/2_inch_lumber.html

3. The costs of different lengths of unplaned and planed pieces of wood at Beardsley's Sawmill in Greene, New York, are given in the table above.

   (*a*) How would you describe mathematically the relationships between the length of the wood and the cost for unplaned and planed products?

   (*b*) Determine equations for the cost as a function of length of unplaned wood and planed wood.

   (*c*) Use your equations from part 3(*a*) to suggest how much this sawmill would charge for 20-ft.-long pieces of unplaned and planed wood.

Commercial Lumber Sizes

Civil Engineering Design Knowledge

The following data is standard reference and size data for commercially available lumber within the USA

Nominal Size	Actual Size	Weight / Foot **
1 X 3	.75 X 2.5	0.47
1 X 4	.75 X 3.5	0.64
1 X 6	.75 X 5.5	1.00
1 X 8	.75 X 7.25	1.32
1 X 10	.75 X 9.25	1.89
1 X 12	.75 X 11.25	2.05
2 X 3	1.5 X 2.5	0.94
2 X 4	1.5 X 3.5	1.28
2 X 6	1.5 X 5.5	2.00
2 X 8	1.5 X 7.25	2.64
2 X 10	1.5 X 9.25	3.37
2 X 12	1.5 X 11.25	4.10
2 X 14	1.5 X 13.25	4.83

Source: Commercial Lumber Sizes, http://www.engineersedge.com/commercial_lumber_sizes.htm

4. In the table at left, *nominal size* and *actual size* mean the same as *unplaned* and *planed*, respectively, as discussed in questions 1–3. The numbers in the weight/foot column represent the weight (in lb.) of 1 ft. of wood.

   (*a*) Determine a rule that can be used to calculate the numbers in the weight/foot column.

   (*b*) Use your rule to calculate the weight of 1 ft. of a 2.5 in. × 15.25 in. planed piece of wood.

"Lumbering Along" answers

1. (*a*) The volume of a 1-ft.-long, unplaned 2 in. × 4 in. piece of wood is 96 in.^{3}. The volume of a 1-ft.-long, planed piece of wood is 63 in.^{3}. Therefore, 33 in.^{3} of wood was removed during the sawing process—about a 34% loss.

(*b*) When $x = 0$, $y = 1$; and when $x = 2$, $y = 0$. Therefore, the graph begins at (0, 1) and ends at (2, 0). Answers for a rough graph will vary but should be consistent with the fact that y is a strictly decreasing function of x from $x = 0$ to $x = 2$. See part 1(*c*) for an accurate graph.

(*c*) The equation for y in terms of x is

$$y = \frac{(2-x)(4-x)}{2 \cdot 4}.$$

The graph of y versus x is shown in **figure 1**. The context of the problem dictates that $0 \le x \le 2$. We include the endpoints, despite realizing that if $x = 2$, the volume is 0, and that if $x = 0$, the wood has not been planed.

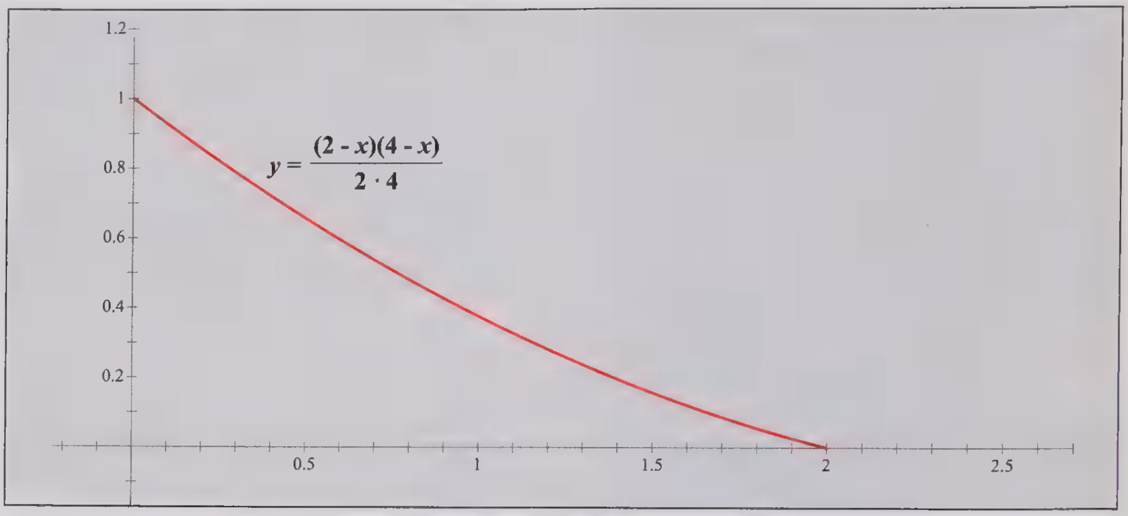


Fig. 1 The graph is parabolic in nature, but the domain is limited to values between 0 and 2.

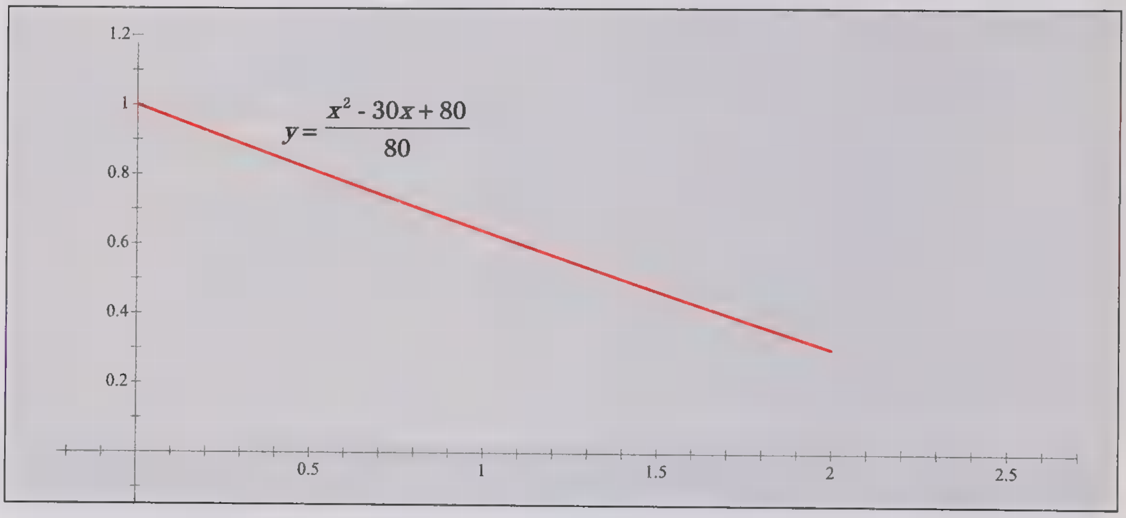


Fig. 2 The domain for the function in part 1(e) is real numbers between 0 and 2.

(*d*) An approximate value of x can be found by finding the point of intersection between the graph in part 1(*c*) and the line $y = 0.5$. As shown below, it can also be found using algebra:

$$\frac{(2-x)(4-x)}{2 \cdot 4} = \frac{1}{2}$$
$$(2-x)(4-x) = 4$$
$$x^2 - 6x + 8 = 4$$
$$(x-3)^2 - 1 = 4$$
$$x = 3 + \sqrt{5}$$
$$\text{or } x = 3 - \sqrt{5}$$

Only one of these values for x lies in the domain of the variable, so $x \approx 0.76$. Thus, half the volume of the wood would be lost if about three-quarters of an inch of wood were shaved off from each side.

(*e*) The equation for y in terms of x can be developed as shown here:

$$y = \frac{2\left[(2-x)(4-x) + 12(2-x) + 12(4-x)\right]}{2(2 \cdot 4 + 2 \cdot 12 + 4 \cdot 12)}$$
$$y = \frac{x^2 - 30x + 80}{80}$$

The graph of y versus x is given in **figure 2**. This graph provides students with an opportunity to see that a portion of a curved graph can resemble a straight line when examined over a short interval (see **fig. 3** for a view of the graph over a larger interval). These graphs could be used as part of an introductory lesson on the tangent to a curve at a point and how the tangent can closely approximate a curve in the neighborhood of a point.

(*f*) An approximate value of x can be found by finding the point of intersection between the graph in part 1(*e*) and the line $y = 0.5$. An algebraic solution is shown below:

$$\frac{x^2 - 30x + 80}{80} = \frac{1}{2}$$
$$x^2 - 30x + 80 = 40$$
$$x^2 - 30x + 40 = 0$$
$$x = \frac{30 \pm \sqrt{740}}{2}$$

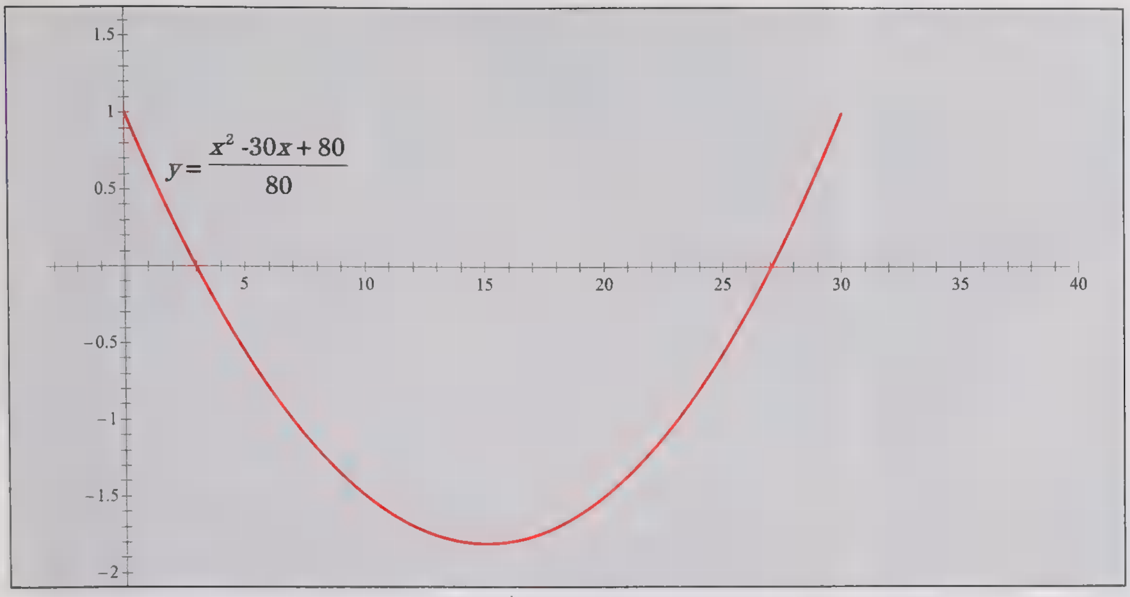


Fig. 3 When viewed over a larger domain, the graph shown in **figure 2** takes on the typical parabolic shape.

Table 1

Side 1	Side 2	Area of Cross Section	Weight	Ratio of Weight to Area of Cross Section
0.75	2.5	1.875	0.47	0.250666667
0.75	3.5	2.625	0.64	0.243809524
0.75	5.5	4.125	1.00	0.242424242
0.75	7.25	5.4375	1.32	0.242758621
0.75	9.25	6.9375	1.69	0.243603604
0.75	11.25	8.4375	2.05	0.242962963
1.5	2.5	3.75	0.94	0.250666667
1.5	3.5	5.25	1.28	0.243809524
1.5	5.5	8.25	2.00	0.242424242
1.5	7.25	10.875	2.64	0.242758621
1.5	9.25	13.875	3.37	0.242882883
1.5	11.25	16.875	4.10	0.242962963
1.5	13.25	19.875	4.83	0.243018868

The two values for x are either more than 28 or around 1.4. Only one value is in the domain. Therefore, half the surface area of the wood would be lost if about 1.4 in. of wood were shaved off each dimension.

2. (*a*) $2(2 + 4) = 12$ is greater than 6 1/4 in. Therefore, a 2×4 piece of wood does not meet the requirements for a handrail in Old Bridge Township.

(*b*) The range of values of x can be found as shown below:

$$2(2-x+4-x) \le 6.25$$
$$12-4x \le 6.25$$
$$x \ge 1.4375$$

The piece of wood will meet the requirements if 1.4375 in. or more (up to 2 in.) of wood is shaved off each dimension.

3. (*a*) As the length increases by the same amount in **table 1** (2 ft.), the increases in the cost of both the unplaned wood and the planed wood appear to be roughly the same, indicating linear relationships.

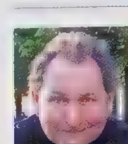
(*b*) Performing linear regression on the data in **table 1** produces the following equations for the unplaned and planed woods, respectively: $y = 0.4425x - 1.086$; and $y = 0.5095x - 1.09$.

(*c*) The equations suggest that this sawmill would charge \$7.76 for a 20-ft.-long piece of unplaned wood and \$9.10 for a planed one.

4. (*a*) Because the weight of the wood depends on the amount after planing, it makes more sense to study the relationship between the weight of a 1-ft. board and the actual size, particularly the area of the cross section. **Table 1** shows that the ratio of the weight to the cross-sectional area is almost the same for every case. Therefore, the weight per foot of a piece of planed wood is approximately equal to 0.24 times the cross-sectional area.

A footnote on the website where these data appear states that the data for the weight is "based on 35 lbs./ft.^{3} actual volume and weight." Ask students to determine whether this statement is consistent with the rule just calculated.

(*b*) The product of $0.24 \cdot 2.5 \cdot 15.25$ is equal to 9.15. So the weight of 1 ft. of this piece of wood is about 9.15 lb. The website gives the weight as 9.27 lb., close to our calculated result.



RON LANCASTER, ron2718@nas.net, is a senior lecturer at the Ontario Institute for Studies in Education of the University of Toronto. His interests include photographs and videos of mathematics, Math Trails, magic, and puns.

Michael H. Koehler

How Many Chips off the Old Block?

The sculpture entitled *Four-Sided Pyramid* by Sol LeWitt (1928–2007) is located in the National Gallery of Art Sculpture Garden on the National Mall in Washington, D.C. Installed in 1999, the sculpture takes on a different appearance depending on light conditions and the location of the sun. The square pyramid consists of rectangular blocks stacked twenty-four levels high. It comprises twenty-three vertical slices of blocks from the front to the back of the pyramid and forty-seven rows of blocks from one corner of the base to the opposite corner. The blocks are rectangular prisms of dimensions $1 \times 1 \times 2$ units; thus, each block can be thought of as consisting of two cubes.

Photograph 1 shows the pyramid from what we will call the front. **Photograph 2** was taken after walking clockwise in a 90° arc around the pyramid. The first question students typically ask when seeing **photographs 1–5** is, How many blocks are in the pyramid? The number of blocks can be determined by discovering patterns in slices of the pyramid. The questions below will consider three different ways to slice the pyramid: (1) vertically from front to back; (2) horizontally from top to bottom; and (3) vertically from side to side.

Mathematical Lens uses photographs as a springboard for mathematical inquiry and appears in every issue of *Mathematics Teacher*. All submissions should be sent to the department editors. For more background information on Mathematical Lens and guidelines for submitting a photograph and questions, please visit http://www.nctm.org/publications/content.aspx?id=10440#lens.

Edited by ***Ron Lancaster,*** *ron2718@nas.net, University of Toronto, ON, Canada, and* ***Brigitte Bentele,*** *brigitte.bentele@trinityschoolnyc.org, Trinity School, New York, NY*

1. (*a*) Consider **photograph 4**, which represents a detail of the front of the pyramid. In the first vertical slice parallel to the plane of the photograph are 4 blocks. In **figure 1**, the faces of these blocks are denoted by the numeral 1. The faces of the blocks in the second vertical slice that can be seen from this view are labeled with the numeral 2. How many blocks are in this second slice? In the slice behind that one?

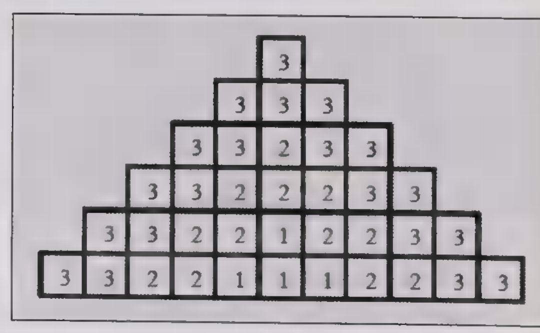
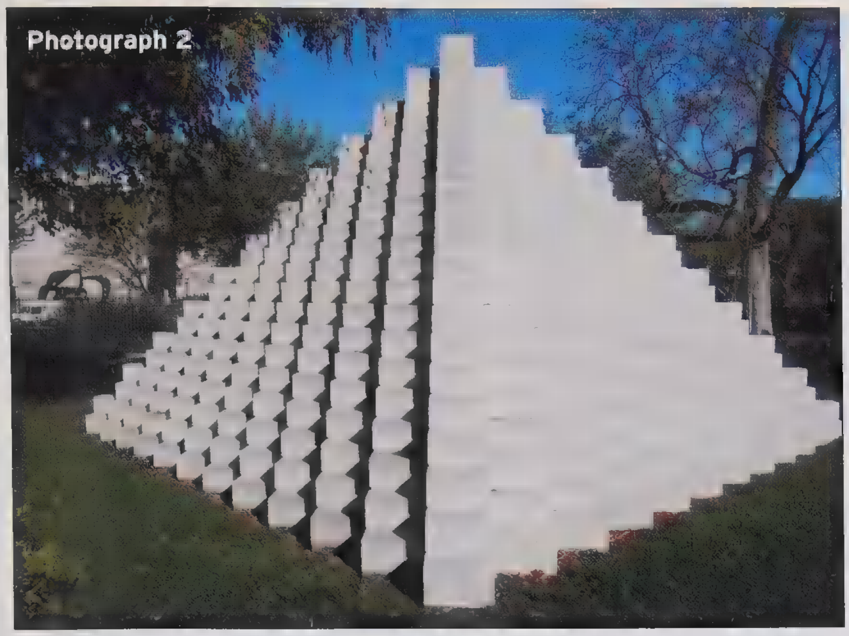


Fig. 1 The numerals indicate which vertical slice the block is in.

   (*b*) How many blocks will be in the *n*th vertical slice?

   (*c*) Using the expression obtained in part (*b*), find the total number of blocks in the pyramid.

2. (*a*) We will now consider horizontal slices. Moving from top to bottom,



determine a pattern that gives the number of blocks for each horizontal slice.

(*b*) Find the general term of the sequence established in part (*a*).

(*c*) Use this sequence to find the total number of blocks in the pyramid.

3. Consider **photograph 2**, taken from a position 90° clockwise from where **photograph 1** was taken. From this view, the pyramid contains forty-seven vertical slices parallel to the plane of the photograph. The blocks making up the pyramid have a different orientation in this view, and the pyramid takes on a different appearance. The first vertical slice taken parallel to the plane of the photograph consists of one block; the face of this block is denoted by the numeral 1 (see **fig. 2**). The faces of the blocks in the second vertical slice that can be seen from this view are labeled with the numeral 2. This numbering pattern continues through seven slices and is also used in **figure 3**, which shows the blocks from a diagonal view.

(*a*) How many blocks are in the second slice? How many blocks are in the third slice? In the fourth slice?

(*b*) How many blocks will be in the *n*th vertical slice?

(*c*) Using the expression in part (*b*), find the total number of blocks in the pyramid.

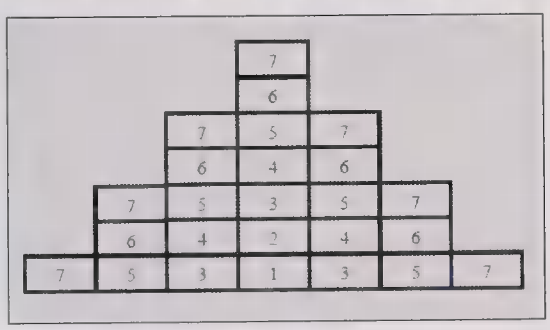


Fig. 2 The first seven levels of blocks are shown from the view in **photograph 2**.

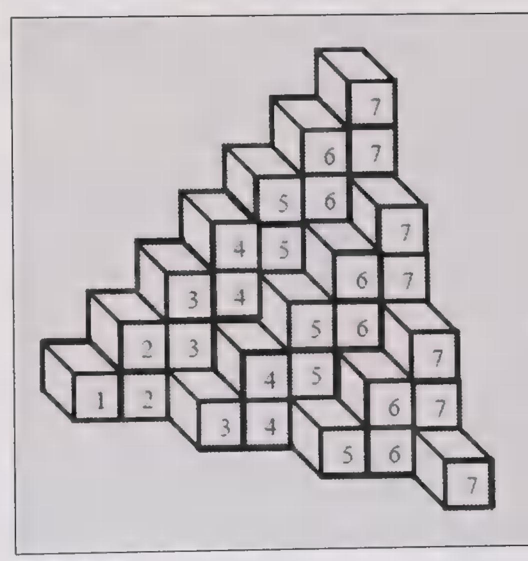


Fig. 3 This figure presents a diagonal view of the blocks shown in **figure 2**.

1. (*a*) The second vertical layer back adds 4 blocks on each of the first two levels and an additional 4 on top in a pattern that duplicates the first layer. Thus, there are 12 additional blocks, for a total of 16. In the third layer, 4 blocks are added on each of the first four levels, with an additional 4 on top, so the third layer has 16 + 20 = 36 blocks. See **figure 1**.

   (*b*) The general term of the sequence is $(2n)^2$, $1 \le n \le 12$ for the front-half slices plus the middle vertical slice and $1 \le n \le 11$ for the back vertical slices.

   (*c*) The total number of blocks is 4624. For the front-half slices plus the middle vertical slice, we have

$$N_{front} = \sum_{k=1}^{n}(2k)^2 = 4\cdot\sum_{k=1}^{n}(k)^2 = 4\cdot\left[\frac{n(n+1)(2n+1)}{6}\right];$$

$$4\cdot\left[\frac{n(n+1)(2n+1)}{6}\right]_{n=12} = 4\cdot\left[\frac{12\cdot13\cdot25}{6}\right] = 2600.$$

   For the back vertical slices, we have

$$N_{back} = 4\cdot\left[\frac{n(n+1)(2n+1)}{6}\right]_{n=11} = 4\cdot\left[\frac{11\cdot12\cdot23}{6}\right] = 2024.$$

   The sum is 2600 + 2024 = 4624. Using the LIST commands on a calculator will also produce a solution.

2. (*a*) **Figure 4** shows the pyramid layers from the top. The top slice contains 1 block, the second slice has 3 blocks, the third slice contains 7 blocks, and the fourth slice has 13 blocks.

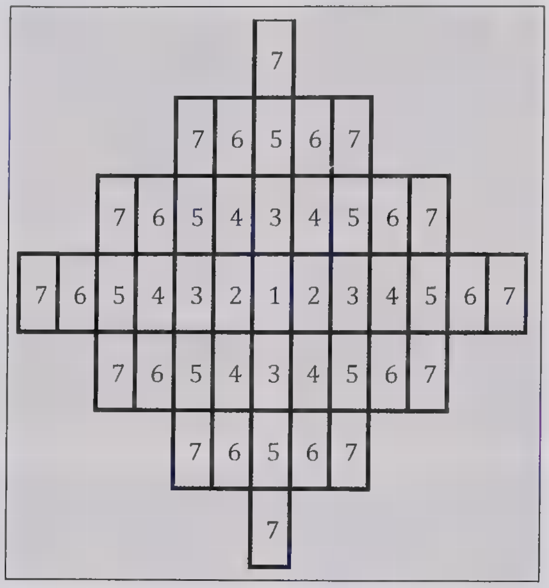


Fig. 4 The first seven levels of blocks from top to bottom are shown; the numeral inside each block indicates the number of the slice. For example, a 3 indicates that the block is in the third slice down.

   (*b*) We will use difference equations to establish an expression for the general term of the sequence. **Table 1** shows the first and second differences of the number of blocks in each level. When the second differences are constant, the relationship is quadratic and takes the form $a_n = An^2 + Bn + C$, where n represents the number of the horizontal slice and a_n represents the number of blocks in that slice. Substituting three values for n and a_n into the equation $a_n = An^2 + Bn + C$ and solving the system of three equations in three unknowns produces the solution (1, –1, 1) for (A, B, C). Thus, $a_n = n^2 - n + 1$.

   (*c*) The total number of blocks is 4624:

$$\sum_{n=1}^{24} n^2 - n + 1 = 4624$$

3. (*a*) The second vertical slice adds 1 block behind the first and 1 additional block on top of this new block, for a total of 2 blocks. The third vertical slice adds 1 additional block on each side and 1 block on top of the previous slice, adding 3 blocks to the 2 in the previous slice, for a total of 5 blocks. The fourth slice repeats the previous slice, adding 1 block on top of each column, for an additional 3 blocks and a total of 8 blocks.

   (*b*) The even slices repeat the previous slice and increase the height by 1 block. The odd slices repeat the previous slice, increase the height by 1 block, and add 1 block to each side of the bottom layer. The sequence of number of blocks for the vertical slices is 1, 2, 3, 5, 8, 13, ... The first differences between terms yield the sequence 1, 3, 3, 5, 5, . . . , whereas the second differences yield the sequence 2, 0, 2, 0, 2, 0, ... Although the second differences are not constant, the fact that the terms alternate between 2 and 0 indicates that the sequence might involve two quadratic alternating sequences. **Table 2** compares the number of blocks with the square of the term number and shows that the number of blocks is $n^2/2$ for even slices and $n^2/2 + 1/2$ for odd slices.

Table 1 First and Second Differences in the Number of Blocks

Level number = n	1	2	3	4	5	6	7
Number of blocks = a_n	1	3	7	13	21	31	43
First difference	2	4	6	8	10	12	
Second difference		2	2	2	2	2	

Table 2 Slices and Blocks

Slice number n	1	2	3	4	5	6	7	8	9
n^2	1	4	9	16	25	36	49	64	81
Number of blocks	1	2	5	8	13	18	25	32	41

We can write an expression

$$\left(\frac{1-(-1)^n}{4}\right),$$

which will return 0 when n is even and 1/2 when n is odd. The general term of the sequence then becomes

$$\frac{1}{2}n^2+\frac{1-(-1)^n}{4}=\frac{1}{4}\left[2n^2+1-(-1)^n\right],$$

$1 \le n \le 24$ for the left-half slices plus the middle vertical slice and $1 \le n \le 23$ for the right-half vertical slices.

(c) The total number of blocks is 4624. For the left-half slices plus the middle vertical slice, we have

$$N_{left}=\sum_{k=1}^{24}\frac{1}{4}\left[2k^2+1-(-1)^k\right]=2456. \quad (1)$$

For the right-half vertical slices, we have

$$N_{right}=\sum_{k=1}^{23}\frac{1}{4}\left[2k^2+1-(-1)^k\right]=2168.$$

Editor's note: For more on mathematics and the art of Sol LeWitt, see Mathematical Lens, *MT* May 2012, vol. 105, no. 9, pp. 652–56.

MICHAEL H. KOEHLER, mkoehler@bluevalleyk12.org, teaches mathematics at Blue Valley North High School in Overland Park, Kansas. His interests include the use of technology for conceptual understanding of mathematics and presenting workshops involving AP Calculus and technology in the mathematics classroom.

Modeling Fuel Efficiency: MPG or GPHM?

The EPA's switch in rating fuel efficiency from miles per gallon to gallons per hundred miles with the 2013 model-year cars leads to interesting and relevant mathematics with real-world connections.

Kevin G. Bartkovich

Since the OPEC oil embargo of the early 1970s, the United States has become ever more conscious of the fuel efficiency of passenger cars. The standard for measuring fuel efficiency in the U.S. has been miles per gallon (mpg), and Americans are constantly buffeted with mpg ratings through new car advertisements, "stickers" on vehicles at car dealerships, and news reports of the latest political debate regarding vehicle emissions. In the midst of all this information, the use of mpg tends to obscure the implications of fuel efficiency standards, leading to widespread misconceptions about the mathematics of mpg.

By modeling fuel efficiency, mathematics teachers have an opportunity to correct mistaken notions while bringing real-world mathematics into the classroom. In particular, we will explore why average mpg is computed not by using the arithmetic mean but by using the harmonic mean. This investigation will lead us to model fuel efficiency with gallons per hundred miles (gphm), and the advantages we find in this new model will illustrate why stickers for the 2013 model-year cars display gphm as well as mpg. As we proceed, we will face counterintuitive results while we explore rich mathematics topics accessible to high school algebra students.

I have used the material presented here in second-year algebra classes

Efficiency:

and mathematical modeling classes for students who have had a year of precalculus. In both settings, I approach the topic as a stand-alone investigation by giving the students the problem set shown in **figure 1**, a customizable copy of which is available online (**www.nctm.org/mt**). We spend a class period working collaboratively through the questions, and those we do not get to in class are assigned for homework. At the next meeting, we present results and finish up the discussion. Although my presentation in this article is necessarily didactic, my role in the classroom is to offer suggestions along the way so that students discover and make sense of the mathematics themselves.

This investigation aligns well with the reasoning and sense making focus of NCTM, particularly number sense. It also supports NCTM's Algebra Standard for grades 9–12, especially the skillful use of mathematical models. Students

are expected to make connections between various real-world numbers and different representations and are asked to generalize their results. Of primary importance, students are encouraged to act as problem solvers throughout the investigation, extending their understanding by working through a succession of problems that build from specific numbers to algebraic expressions to the interpretation of their results (NCTM 2000, 2009).

THE BRILLIANCE OF CASH FOR CLUNKERS

We begin with question 1 from the problem set, which concerns how increasing mpg ratings affects fuel economy:

1. Which of the following transactions do you think results in the greatest gasoline savings?

MPG vs. GPHM

1. Discuss in small groups: Without going into the calculations, which of the following transactions do you think results in the greatest gasoline savings?
   (*a*) Trading in a car that gets 10 mpg for a car rated at 11 mpg
   (*b*) Trading in a car that gets 22 mpg for a car rated at 26 mpg
   (*c*) Trading in a car that gets 38 mpg for a car rated at 50 mpg

2. In each case in problem 1, calculate the gasoline savings that would result from a year's driving—typically, 12,000 miles. How much would this save in annual carbon dioxide emissions if burning 1 gallon of gasoline emits 20.4 pounds of CO_2?

3. Fill in the blanks so that all three transactions result in about the same gasoline savings. What is the savings in a year of driving?
   (*a*) Trading in a car that gets 10 mpg for a car rated at 11 mpg
   (*b*) Trading in a car that gets 16.5 mpg for a car rated at _____ mpg
   (*c*) Trading in a car that gets _____ mpg for a car rated at 50 mpg

4. (*a*) Fill in the following table for miles per gallon vs. gallons per hundred miles.

MPG	GPHM
10	
20	
25	
	2.5
50	
80	
	1

   (*b*) Sketch a graph of mpg vs. gphm. What type of function is represented by this graph?
   (*c*) If you want to calculate gasoline savings in a comparison of two cars, which rating is more helpful—mpg or gphm?

5. When you drive to work, your hybrid car goes uphill for 20 miles, and you notice that the fuel efficiency during that time is 20 mpg. Returning home, you drive downhill for the same 20 miles, and your fuel efficiency is 100 mpg. What is your average mpg for the round-trip? Fill in the following chart.

Trip	Miles	MPG	Gallons Used	GPHM
Uphill	100	20		
Downhill	100	100		
Total	200			

6. (*a*) If you have two cars, one of which is rated 14 mpg and the other 36 mpg, what is your average mpg, assuming that you drive each car the same number of miles?
   (*b*) If you have two cars, one of which is rated 5 gphm and the other 2 gphm, what is your average gphm?

7. Which gives the greater fuel savings— doubling the mpg of your vehicle from 10 mpg to 20 mpg or doubling from 25 mpg to 50 mpg?

8. What strategy would have a greater overall effect on carbon emissions—raising the efficiency of the 50-mpg vehicles by 10% or raising the efficiency of an equal number of 20-mpg vehicles by 10%?

9. (*a*) Discuss the relative merits of the mpg rating and the gphm rating. In which situation is one more helpful than the other?
   (*b*) Fill in the blanks in the following statement and then explain your answer. "To know your gas savings when comparing cars, _____ can be subtracted; _____ cannot."

10. This is a generalization of problem 6.
   (*a*) Derive a formula for finding a family's average mpg if the family has two vehicles that have differing mpgs but that are driven the same number of miles. Your answer is the *harmonic mean* of the two mpg numbers. This answer contrasts with an *arithmetic mean*, the sum of two numbers divided by 2, which was how we averaged gphm values in problem 6.
   (*b*) The EPA calculates combined mpg by assuming 55% city driving and 45% highway driving. What is the combined mpg for a vehicle that is rated 23 mpg city and 34 mpg highway?

Fig. 1 This problem set can be used in a second-year algebra class.

Table 1 Calculations of Gas Savings for Problem 1

MPG	GPM	Gas per Year (gallons)	Difference (gallons)
10	0.1000	1200	109
11	0.0909	1091	
22	0.0455	545	83
26	0.0385	462	
38	0.0263	316	76
50	0.0200	240	

Table 2 MPG and the Corresponding GPHM

MPG	GPHM
10	10.0
20	5.00
25	4.00
40	2.50
50	2.00
80	1.25
100	1.00

(*a*) Trading in a car that gets 10 mpg for a car rated at 11 mpg
(*b*) Trading in a car that gets 22 mpg for a car rated at 26 mpg
(*c*) Trading in a car that gets 38 mpg for a car rated at 50 mpg

When I present this question to my students, I press them to answer quickly using only mental arithmetic and their mathematical intuition. Nearly all guess answer (*c*), reasoning that it reflects the greatest increase in mpg, even on a percentage basis, so it must represent the most gasoline saved. Occasionally, one student suspects that some sort of trickery is afoot and will go with answer (*a*) because it seems the least likely of the three choices. At this point, we do not critique answers; rather, students set about working on the problem in small groups.

How do we calculate the gasoline savings in each of these transactions? The mpg rating tells us how far we can drive on a gallon of gas; therefore, to determine the amount of gas used, we need to know the number of miles driven. For example, if a car that is rated 20 mpg is driven 1000 miles, then the amount of gas used is 1000/20, or 50 gallons. In class, we use the average distance that a car is driven in a year, which we estimate at 12,000 miles. Answer (*a*) indicates that a car that gets 10 mpg will use an amount of fuel given by

$$\frac{12{,}000 \text{ miles}}{10 \text{ mpg}},$$

which equals 1200 gallons. As a side note, here is an opportunity to encourage students to carry the units through the calculations. Thus, the units of miles in the numerator and denominator cancel, and the per gallon unit in the denominator is inverted to become gallons in the final answer.

Notice that to calculate fuel consumption, we divided by 10 mpg, a step that is equal to multiplying by its reciprocal, 0.1 gallons per mile. Rather than dividing miles by mpg, we can express this calculation also as miles *times* gallons per mile (miles • gpm). Using a similar strategy with the 11 mpg car, we find that its gpm rating equals 1/11 mpg, or about 0.0909 gpm. The gas used per year is thus 12,000 miles • 0.0909 gpm, or about 1091 gallons. The difference of the results of these two calculations gives us a gas savings in answer (*a*) of about 109 gallons per year.

The results of similar calculations for the other two trades are shown in **table 1**. We find that the savings in answer (*b*) is about 83 gallons per year and that the savings in answer (*c*) is about 76 gallons per year. Overall, these three calculations yield numbers that are counterintuitive: The small change from 10 mpg to 11 mpg results in the greatest savings in fuel consumption, and the greatest change in mpg, from 38 to 50, results in the least savings of the three choices. An mpg of 50 sounds a lot better than 38, and it is, but an mpg of 11 compared with 10 is even better. A small increment at the low end of mpg makes a bigger difference in fuel savings than a much larger increase at the high end of mpg. This fact was part of the rationale behind the federal government's Cash for Clunkers program in the summer of 2009, whereby consumers could trade in old cars rated at less than 18 mpg and receive a rebate on a new car with a rating of at least 22 mpg.

CONSERVATION REQUIRES AN UNDERSTANDING OF MPG VS. GPHM

How can we make sense of these counterintuitive results? In **table 1**, we saw that what really mattered was gpm. Even though the numbers were scaled by a factor of 12,000, the differences in gpm determined the greatest fuel savings. Now let's take a look at how gpm relates to mpg but with a slight adjustment. To make the numbers not so small, we will use gallons per hundred miles (gphm), a commonly used measurement that equals gpm multiplied by 100.

Table 2 gives the values for gphm that correspond to selected mpg values. (This calculation is what students are expected to do for question 4 in

the problem set.) For example, if a car has a rating of 10 mpg, then to drive 100 miles would require 10 gallons of gas. At the other end of the table, a car rated at 100 mpg would need only 1 gallon of gas to go 100 miles. In general, the entries in **table 2** are calculated by taking the reciprocal of mpg, which yields gpm, and then multiplying by 100 to get gphm, which can be written as

$$\text{gphm} = \frac{100}{\text{mpg}}.$$

A graph of this relationship is shown in **figure 2**.

Referring to the numbers in **table 2**, the equation for gphm, and the graph in **figure 2**, we see that the relationship between mpg and gphm is reciprocal, also known as an *inverse variation*. The part of the graph to the left, where mpg is low, has a steep decline, meaning that a small increase in mpg leads to a big decrease in gphm. Conversely, the graph flattens out as it moves to the right, where mpg is high, meaning that a relatively large increase in mpg will have little effect on the gphm. As we move up the mpg scale, we require a greater increase to produce the same effect on gas savings. This result tells us that an increase at the high end of the mpg scale is not where we will get the greatest impact on reducing fuel consumption. This is not to say that consumers should not purchase a 40 mpg car instead of a 30 mpg car if given the chance. Rather, it means that the greatest impact on fuel consumption and carbon emissions will result from focusing conservation efforts on the "gas guzzlers" at the low end of the mpg scale.

THE HARMONIC MEAN AND THE TRICKY NATURE OF MPG AVERAGING

A second property of the mpg rating that is more complicated than it seems at first glance is how an average mpg is calculated. We make this calculation in the following question (adapted from problem 5 in the problem set):

5. When you drive to work, your hybrid car goes uphill for 20 miles, and you notice that the fuel efficiency during that time is 20 mpg. Returning home, you drive downhill for the same 20 miles, and your fuel efficiency is 100 mpg. What is your average mpg for the round-trip?

Even after the previous discussion and counterintuitive results from question 1, many students will still guess the arithmetic average of 20 and 100, or 60 mpg—the only type of average they are familiar with, given their mathematics background. There are always a few students, however, who are skeptical of this answer and suggest that things are a bit more complicated than that. At this point, we try to answer the question by examining what we mean by average mpg.

For any trip, the average mpg is the ratio of total miles traveled to the gallons of gas used, as shown in the equation

$$\text{average mpg} = \frac{\text{total distance}}{\text{total gas consumption}}.$$

We know the distance for each leg of the round-trip, but we need to determine the gas consumption for each leg. As in question 1, we can calculate gas consumption by dividing the miles traveled by the mpg rating. On the uphill drive, we will use gas in the amount of 20 miles/20 mpg, which is 1 gallon. On the downhill return, we will use the quantity 20 miles/100 mpg, which is 0.2 gallons. The overall average mpg for the round-trip is therefore 40 miles/1.2 gallons, or about 33 mpg. Clearly, our answer is *not* the arithmetic average of 20 and 100.

We can generalize the calculation of average mpg (question 10[*a*] in the problem set) by letting d miles represent the total distance driven; thus, each leg of the trip is $(1/2)d$ miles. If the first leg has an mpg rating of m_1, then the amount of gas used can be found by dividing distance by mpg, or $(1/2)d/m_1$. Likewise, if the second leg has an mpg rating of m_2, the amount of gas used during the second part can be found by $(1/2)d/m_2$. When we substitute these expressions into the quotient of total distance divided by total gas consumption, we have the equation

$$\text{average mpg} = \frac{d}{\frac{\frac{d}{2}}{m_1} + \frac{\frac{d}{2}}{m_2}}.$$

We factor the variable d from the numerator and both terms in the denominator and reduce, resulting in

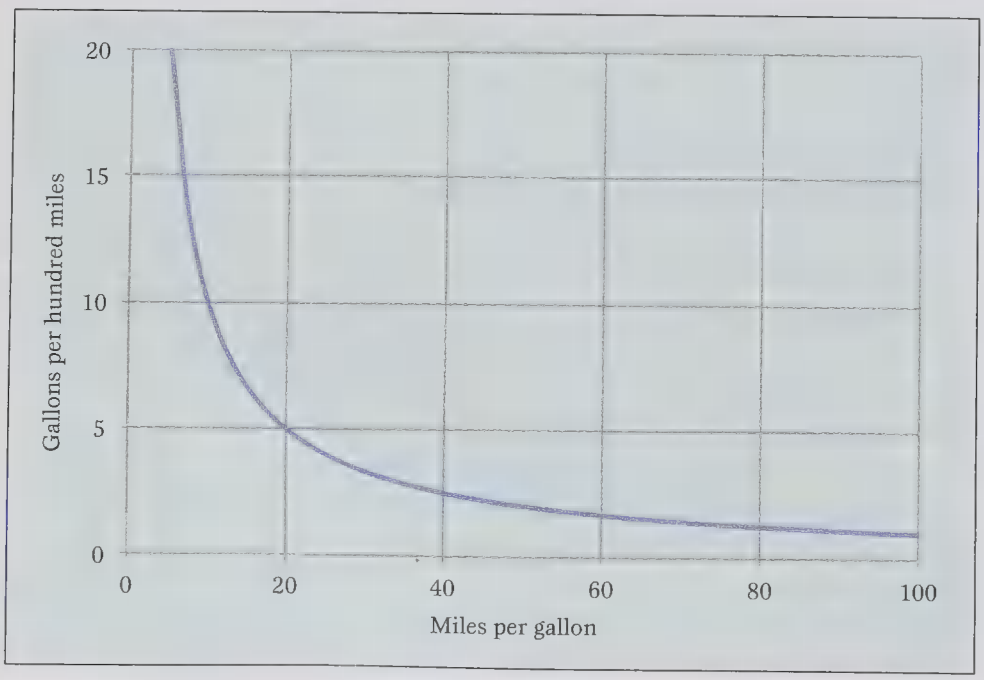


Fig. 2 The graph illustrates the inverse relationship between gphm and mpg.

$$\text{average mpg} = \frac{1}{\left(\frac{1}{2}\right)\left(\frac{1}{m_1} + \frac{1}{m_2}\right)}.$$

The expression on the right is known as the *harmonic mean* of m_1 and m_2. The harmonic mean can be described as the reciprocal of the arithmetic mean of the reciprocals. Now we have shown that the average mpg for a trip that consists of two equal distances with two different mpg ratings is the harmonic mean of the mpg ratings.

REINFORCING THE HARMONIC MEAN

The following questions (problem 6 in the problem set) highlight the distinction between mpg and gphm:

6. (*a*) If you have two cars, one of which is rated 14 mpg and the other 36 mpg, what is your average mpg, assuming that you drive each car the same number of miles?
   (*b*) If you have two cars, one of which is rated 5 gphm and the other 2 gphm, what is your average gphm?

The answer to part (*a*) can be calculated in the same manner as problem 5, and the answer is the harmonic mean of 14 and 36, which is about 20 mpg. Notice that the harmonic mean is closer to the lower mpg rating as compared with 25 mpg, the arithmetic mean of 14 and 36. Even after the analysis of problem 5, some students will still give the arithmetic mean as their answer. The use of the harmonic mean is unfamiliar to them, so this concept will continue to need reinforcement.

To answer part (*b*), because the units are gallons per hundred miles, let's suppose that we drive each car 100 miles. The first one, which rated 5 gphm, will consume 5 gallons of gas; the second one, which rated 2 gphm, will use 2 gallons of gas. The total gas consumption will be 7 gallons, and the total distance traveled will be 200 miles. The result is a ratio of 7 gallons per 200 miles, which is the same as 3.5 gallons per 100 miles; therefore, the average rating for the two cars is 3.5 gphm. Our answer is simply the arithmetic average of 5 and 2.

The analysis of parts (*a*) and (*b*) reveals a crucial advantage of gphm over mpg: gphm ratings can be averaged in the familiar arithmetic way, but mpg ratings cannot. The answer to part (*a*) requires the harmonic mean, which is not a simple calculation. In contrast, finding the arithmetic mean in part (*b*) is straightforward, and most algebra students can do this mentally.

Misconceptions about how to calculate average mpg extend to what is shown on a vehicle's sticker on the showroom floor. Two mpg ratings are always given, one for city driving (the lower number) and one for highway driving (the higher number). Then there is often a third number between the two that is called the *combined mpg*, which is the subject of the following question (problem 10[*b*] in the problem set):

10. (*b*) The EPA calculates combined mpg by assuming 55% city driving and 45% highway driving. What is the combined mpg for a vehicle that is rated 23 mpg city and 34 mpg highway?

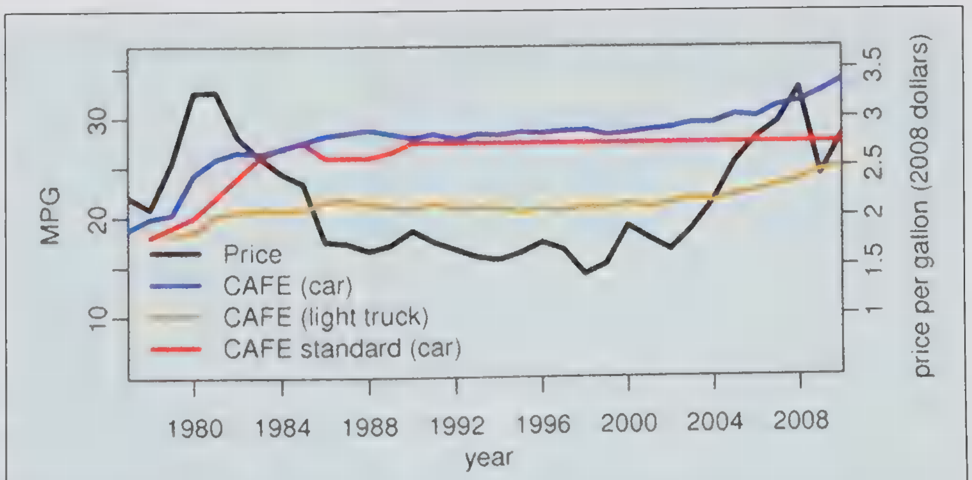
Recall that we found the average mpg in problem 6(*a*) by assuming the same driving mileage for each car. In the harmonic mean formula, the terms $1/m_1$ and $1/m_2$ are both multiplied by 1/2, or 0.5. In the combined mpg, we need to weight the two mpg ratings unequally, one by 0.55 and the other by 0.45. This understanding allows us to calculate the combined mpg using a weighted harmonic mean, as in

$$\text{combined mpg} = \frac{1}{\frac{0.55}{23} + \frac{0.45}{34}},$$

which equals approximately 27 mpg.

THE CAFE STANDARD AND THE MPG ILLUSION

The minimum average mpg for new cars, known as the CAFE (Corporate Average Fuel Efficiency) standard, has been set by the EPA since 1975. From 1990 to 2010, the CAFE standard remained unchanged, at 27.5 (see **fig. 3**). Since 2010, the CAFE standard has been rising because the EPA has mandated an increase of 50% by 2018. The rationale for increased fuel efficiency has two main aspects: (1) to decrease oil consumption; and (2) to reduce carbon emissions. The first is often cited as a way to reduce U.S. dependency on imported oil, whereas the second is linked to greenhouse gas



Source: Wikipedia Commons, 2007, http://en.wikipedia.org/wiki/File:CAFEStandard2.png

Fig. 3 The graph of the CAFE standard between 1990 and 2010 shows little change.

How is a manufacturer's CAFE determined for a given model year?

A manufacturer's CAFE is the fleetwide average fuel economy. Separate CAFE calculations are made for up to three potential fleets: domestic passenger cars, imported passenger cars, and light trucks. The averaging method used is referred to as a *harmonic mean*. . . . The numerical example below illustrates the process. Assume that a hypothetical manufacturer produces four light truck models in 2004, where mpg means miles per gallon and gvwr means gross vehicle weight rating, measured in pounds:

Model	MPG	GVWR	Production Volume
Vehicle A	22	3,000	130,000
Vehicle B	20	3,500	120,000
Vehicle C	16	4,000	100,000
Vehicle D	10	8,900	40,000

Because vehicle D exceeds 8,500 gvwr, it is excluded from the calculation. Therefore, the manufacturer's light truck CAFE is calculated as follows:

$$\frac{\text{total production volume}}{\frac{\text{no. vehicle } A}{\text{mpg } A} + \frac{\text{no. vehicle } B}{\text{mpg } B} + \frac{\text{no. vehicle } C}{\text{mpg } C}} = \text{average light truck fleet fuel economy}$$

$$\frac{350{,}000}{\frac{130{,}000}{22} + \frac{120{,}000}{20} + \frac{100{,}000}{16}} = 19.27 \text{ mpg}$$

The light truck CAFE standard for the 2004 model year is 20.7 mpg; therefore, the manufacturer is not in compliance.

Source: Adapted from U.S. NHSTA, "CAFE," http://www.nhtsa.gov/cars/rules/cafe/overview.html

Fig. 4 The CAFE is the fleetwide average of fuel efficiencies.

emissions and climate change. How much difference will a 50% increase in mpg make in fuel consumption (and the related carbon emissions)?

Our previous work with mpg leads us to suspect that this 50% increase will not necessarily decrease fuel consumption and the accompanying carbon emissions by 50%. If, for example, we assume that all new vehicles were rated the same as the CAFE standard, we can examine the effect of the mpg increase on gphm. An increase of 50% is accomplished by multiplying mpg by 1.5, and because the relationship between mpg and gphm is reciprocal, this step is equivalent to multiplying gphm by 1/1.5, or 2/3. A two-thirds multiplier yields a 33% reduction, so there is our answer: A 50% increase in mpg leads to a 33% reduction (not 50%) in fuel consumption and associated carbon emissions. Further, by 2025 the CAFE standard will double to around 55 mpg, which by this same reciprocal relationship will reduce carbon emissions for new cars by 50%.

The real-world situation will be much more complicated because new cars will have various mpg ratings scattered above and below the CAFE standard (see **fig. 4**). Even so, our simplified model shows that the percentage change in mpg does not automatically equate with the percentage reduction in gas consumption and carbon emissions; in fact, it overestimates the reductions. In addition, regardless of what new CAFE standards are implemented, they will apply only to new cars; thus, many older, low-mpg vehicles will still be on the roads.

MATHEMATICS AND SUSTAINABILITY

These investigations make a strong case for a shift to gphm as a standard rating for passenger vehicles. In fact, as of the 2013 model year, the EPA has mandated that new car stickers display the gphm rating, although the mpg rating is still more prominent (U.S. Environmental Protection Agency n.d.). By knowing the gphm rating of two vehicles, we can easily make a comparison and calculate fuel savings by simple subtraction. The mpg does not give us these data and in fact obscures the savings, as problem 1 demonstrates. Further, averaging mpg ratings is counterintuitive, whereas averaging gphm ratings is straightforward. Finally, increases in mpg are misleading in relation to fuel savings, especially at the higher values; mpg measures fuel efficiency and thus does not convey the same information as gphm, which measures fuel consumption.

The beauty of this topic is its high level of interest and relevance for students as they consider purchasing and owning their own car. In addition, environmental mathematics and the mathematics of sustainability will continue to be relevant issues for students throughout their adult lives. By bringing this material into the classroom, we can help students understand mpg and gphm, and we can do some interesting mathematics along the way.

BIBLIOGRAPHY

Bogomolny, Alexander. n.d. "Averages, Arithmetic, and Harmonic Means." http://www.cut-the-knot.org/arithmetic/HarmonicMean.shtml.

Larrick, Richard. 2008. "The MPG Illusion." http://www.mpgillusion.com/.

National Council of Teachers of Mathematics (NCTM). 2000. *Principles and Standards for School Mathematics.* Reston, VA: NCTM.

———. 2009. *Focus in High School Mathematics: Reasoning and Sense Making.* Reston, VA: NCTM.

U.S. Department of Energy. n.d. "How Can a Gallon of Gasoline Produce 20 Pounds of Carbon Dioxide?" http://www.fueleconomy.gov/feg/contentincludes/co2_inc.htm.

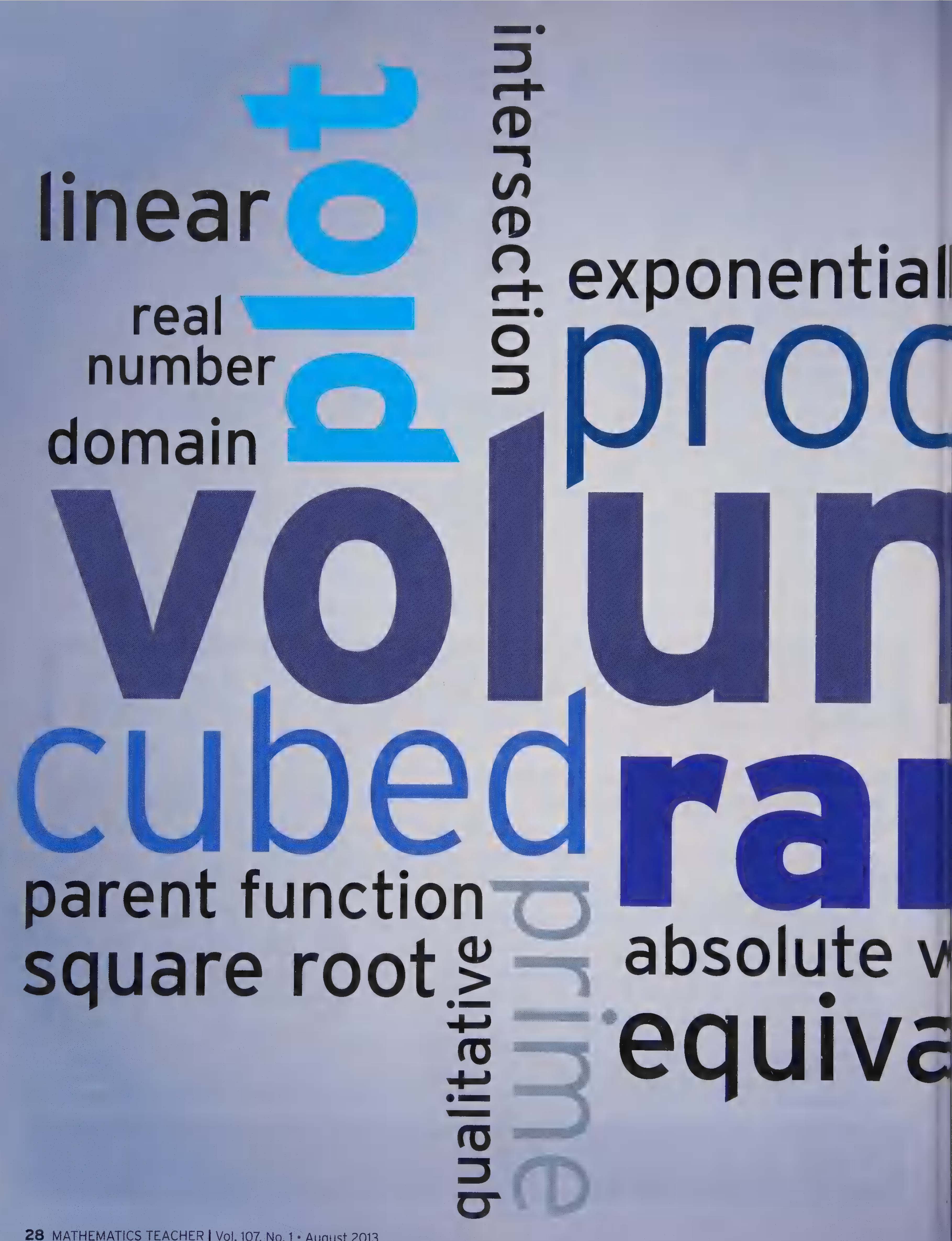
U.S. Environmental Protection Agency. n.d. "Gasoline Vehicle Label." http://www.epa.gov/otaq/carlabel/gaslabel.htm.

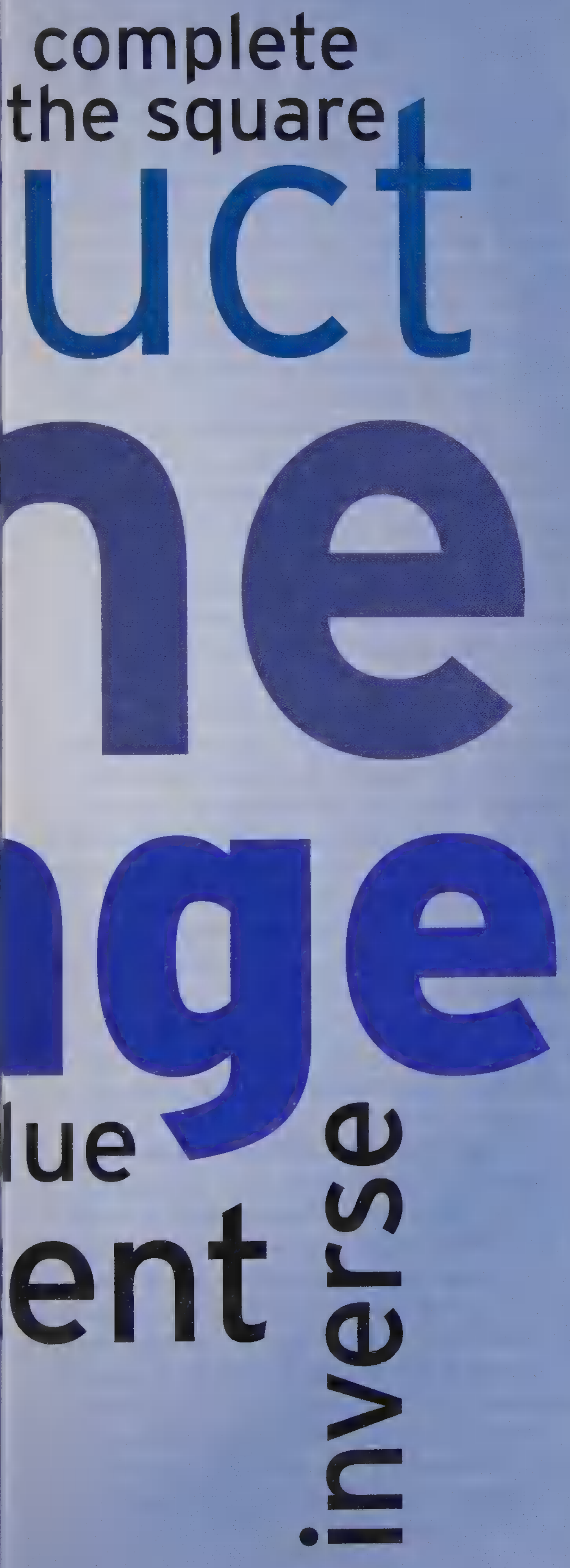
U.S. National Highway Traffic Safety Administration. n.d. "CAFE—Fuel Economy." http://www.nhtsa.gov/fuel-economy/.

KEVIN G. BARTKOVICH, kbartkovich@exeter.edu, is a mathematics instructor at Phillips Exeter Academy in Exeter, New Hampshire. He teaches a problem-based curriculum and is interested in the mathematics of sustainability. Previously, he spent ten years in rural Uganda, where he established a boarding secondary school in Bundibugyo District.

more4U

For an explanation of the CAFE standard, download one of the free apps for your smartphone and then scan this tag to access **www.nctm.org/mt041**. For a customizable version of figure 1, go to **www.nctm.org/mt**.


linear
plot
intersection
exponential
real number
domain
cubed
parent function
square root
qualitative
prime



For ELLs: Vocabulary beyond the Definitions

A classroom teacher discusses ambiguities in mathematics vocabulary and strategies for ELL students in building understanding.

Nancy S. Roberts and Mary P. Truxaw

"But where is the exponent?" Jorge, a tenth-grade English language learner (ELL), asked me while I (co-author Roberts) was talking about the formula for the area of a parallelogram. After much confusion on my part, Jorge said, "Last year you said that the base was the number in a power that was not the exponent. I don't see the exponent, so I don't know where the base is." Aha! I *had* said something like that in algebra class the previous year. However, I had never thought about the two different uses of the word *base* within mathematics.

Although I knew some of the challenges faced by ELL students learning mathematics vocabulary, I had never considered that mathematics, known for its precision, would include ambiguity *within* its vocabulary. In fact, the sixth Standard for Mathematical Practice within the Common Core State Standards for Mathematics (CCSSM) relates to attending to precision: "Mathematically proficient students" need to "communicate precisely to others" and "try to use clear definitions in discussion with others and in their own reasoning" (CCSSI 2010, p. 7).

I thought about Jorge. He had been confident enough and had the language skills to ask for clarification; many ELL students might not. If I had not recognized the connection to my earlier use of mathematics vocabulary, where would this

confusion have led? How would I have uncovered it? How would Jorge's confusion have impeded mastering important mathematical practices or communicating precisely? These questions and others led to my investigation into the role of vocabulary development in helping ELL students be successful in mathematics, specifically in first-year algebra.

As I considered the importance of supporting ELL students' mathematics vocabulary, I asked myself a question that would likely arise for many mathematics teachers: "Do I have time to spend on vocabulary development?" Jorge helped me recognize that I had to ask myself a different question: "Can I afford *not* to spend time on vocabulary development?" Many vocabulary strategies that have worked for my students do not add much additional time and enhance not only vocabulary but also the mathematics.

CHALLENGES OF MATHEMATICS VOCABULARY FOR ELLS

Although mathematics language is much more than just learning vocabulary (Moschkovich 1999, 2002; NCTM 2000), vocabulary development is still central to learning to read, write, speak, listen to, and make sense of mathematics (CCSSI 2010; Heinze 2005). I will focus specifically on helping ELL students build better understanding of algebra through vocabulary, sharing outcomes of my own learning about mathematics vocabulary and strategies that worked for my students and me.

Mathematics vocabulary may be more difficult to learn than other academic vocabulary for several reasons.

Table 1 Math Usage vs. Everyday Usage

Vocabulary Word	Mathematics Usage	Everyday Usage
volume	Amount of space	Noise level
product	Result in multiplying	Item produced in manufacturing
plot	To graph a point	A piece of land to build a house
cubed	Raised to the third power	A type of steak or a way to cut vegetables
range	Numerical difference between two values	Stove top
prime	Prime number	Prime rib, prime time

Source: Adams (2003), p. 789

Table 2 Homonyms and Similar Sounding Words

whole – hole two – to – too tenths – tents	eight – ate symbol – cymbal half – have	sum – some sides – size real – reel

Source: Adams (2003), p. 790–91

1. Definitions are filled with technical vocabulary, symbols, and diagrams (Pimm 1987). Teachers need to explicitly help students make sense of this new language (Schlepegrell 2007).
2. Many mathematics concepts can be represented in multiple ways. At least thirteen different terms can mean subtraction (Echevarria, Vogt, and Short 2010; Heinze 2005). Multiplication can be indicated in many ways: "2 times 3," "2 multiplied by 3," and "the product of 2 and 3." To add to the confusion, some words may have similar connotations but vastly different technical meanings—for example, "3 multiplied by 10" and "3 increased by 10" (Heinze 2005).
3. Many mathematics words have multiple meanings. A *quarter* may refer to a coin or a fourth of a whole. Students must learn that the same word in different situations has different meanings, such as asking for a quarter while at a vending machine or while eating a pizza (Moschkovich 2002).
4. The overlap between mathematics vocabulary and everyday English (Kotsopoulos 2007; Moschkovich 2002) is problematic (see **table 1**). The word *product*, for instance, has meaning in everyday English that is completely different from its very specific mathematical meaning.
5. Homonyms and words that sound similar can confuse (Adams 2003). See **table 2** for a partial list.
6. Similarity to native language words can add more confusion. Although these similarities may sometimes be helpful—as when cognates have similar sounds and similar meanings—similarities can also contribute to confusion. For example, the Spanish word for *quarter* is *cuarto*, which can mean "a quarter of an hour"; *quarter* could also mean a room in a house, as in the English usage "your living quarters" (Moschkovich 1999, 2002).

Clearly, vocabulary is an important issue in mathematics classrooms, especially for ELL students.

TEACHING METHODS AND STRATEGIES

A selection of strategies for supporting students' development of mathematics vocabulary and examples of how to use them follow. Suggestions illustrate vocabulary support within an algebra unit but could be adapted for other topics. Two tools that will be highlighted are word walls—organized collections of words displayed in the classroom to support vocabulary development—and graphic organizers—visual charts and representations designed to organize student learning. We will also look at ways in which these tools can encompass vocabulary strategies.

Table 3 Sample Algebra Vocabulary

absolute value	binomial	coefficient	complete the square	conjunction	derive
domain	equivalent	exponential	function	intersection	interval
inverse	linear	monomial	parabola	parent function	piecewise
polynomial	qualitative	quadratic	radicand	range	rational
real number	regression	solution	square root	trinomial	variable

Source: CCSSI (2010), pp. 52–71

Develop a Vocabulary List

Begin by developing a vocabulary list for the unit. **Table 3** shows samples of mathematics vocabulary from the Common Core State Standards for algebra (CCSSI 2010). Along with traditional algebra terms, include vocabulary to support challenges for ELLs, as described earlier (e.g., *symbol* and *whole*). Scaffolding such as word walls and graphic organizers will increase vocabulary usage while reducing cognitive load and stress (Echevarria, Vogt, and Short 2010).

Preteach and Assess

At the start of a unit, it is beneficial to trigger and assess prior knowledge, review previously learned vocabulary, and preteach new vocabulary. Preteaching vocabulary words requires explicit teaching of definitions, pronunciation, and word parts (Paulsen 2007).

Word Walls

One strategy for stimulating and assessing prior knowledge is a word wall. At the beginning of the unit, display all the vocabulary for the unit to act as an *anticipation guide*, a strategy used during preteaching to stimulate interest in a topic and give students a preview of what is to come. One way to use a word wall as a preassessment tool and as the trigger on the first day of a unit is to include a word that does not belong. Then ask small groups to pick out the word and describe why it does not belong. In a graphing unit, for example, the word wall could include the term *scientific notation* along with graphing words such as *slope*, *y-intercept*, *ordered pair*, *xy-intercepts*, and so on. (The nonconforming word would later be removed from the word wall.)

Another way to use word walls for preassessment is to have students organize the words into groups and give reasons for their choices. Words relating to a unit on exponents might be *base*, *exponent*, *denominator*, *numerator*, *polynomial*, *monomial*, *binomial*, *trinomial*, *power*, *reciprocal*, *coefficient*, and *factor*. One student might group *denominator*, *numerator*, and *reciprocal* as words related to fractions; another student might group *base*, *exponent*, and *power* as words describing a power. Listening to discussions provides interactive forms of preassessment. Moreover, student explanations provide opportunities to foster CCSS mathematical practices—for example, communicating precisely to others and constructing viable arguments.

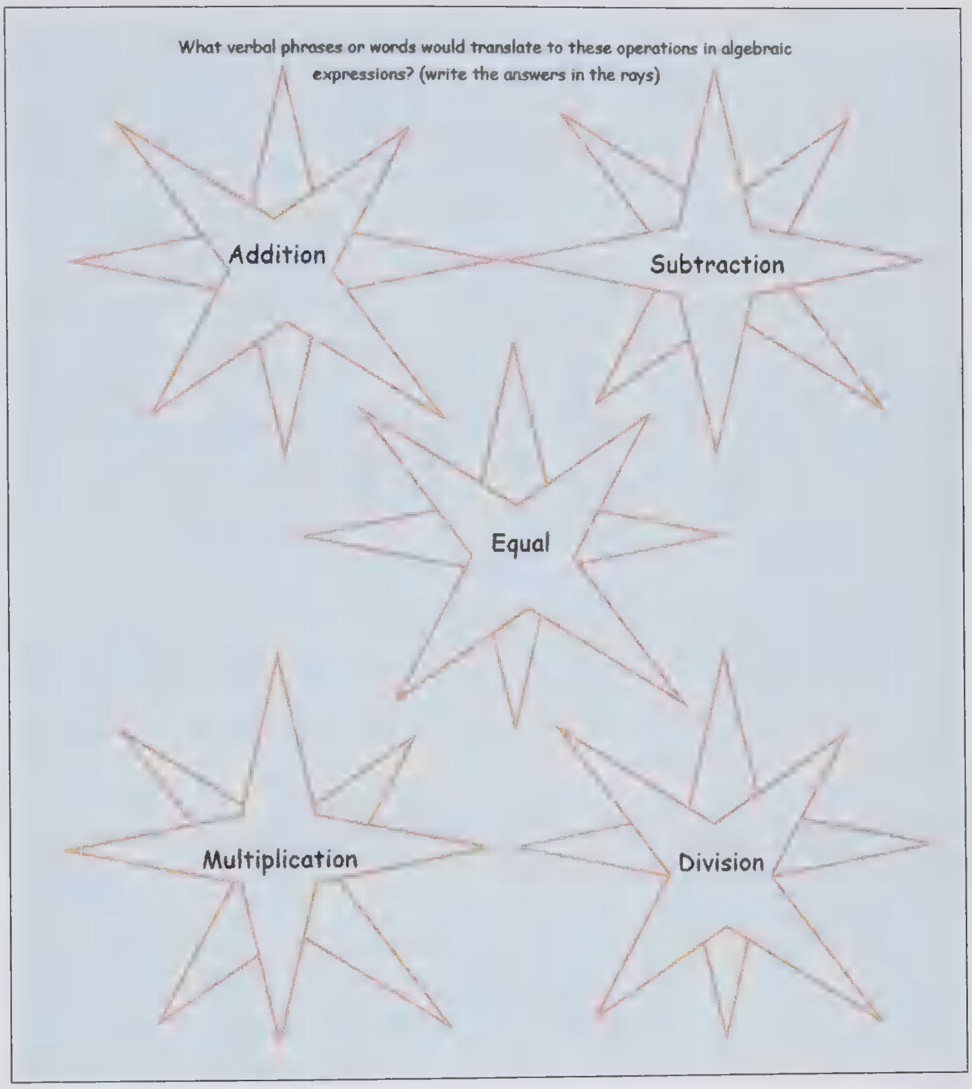


Fig. 1 The points of the stars provide space for students to write phrases that mean the same thing.

Graphic Organizers

Graphic organizers can be useful for activating and assessing students' prior knowledge, organizing different ways to express basic mathematical concepts, and organizing vocabulary for long-term retention. One organizer includes eight-sided stars with words for arithmetic operations and equality (see **fig. 1**). Working with partners, students list

words that could be used for each operation. Then they add to their lists by comparing these in small groups. Finally, the class as a whole reviews the words. This class review is a time to make connections to the mathematical concepts, to address misconceptions, and to include words and phrases that are often confusing—for instance, "4 less than x" to mean "x minus 4."

Teach and Reteach

Researchers have provided many suggestions for explicitly teaching and reteaching vocabulary (see, e.g., Adams 2003; Gee 1996; Moschkovich 2002; Paulsen 2007). The focus here will be on word walls and graphic organizers.

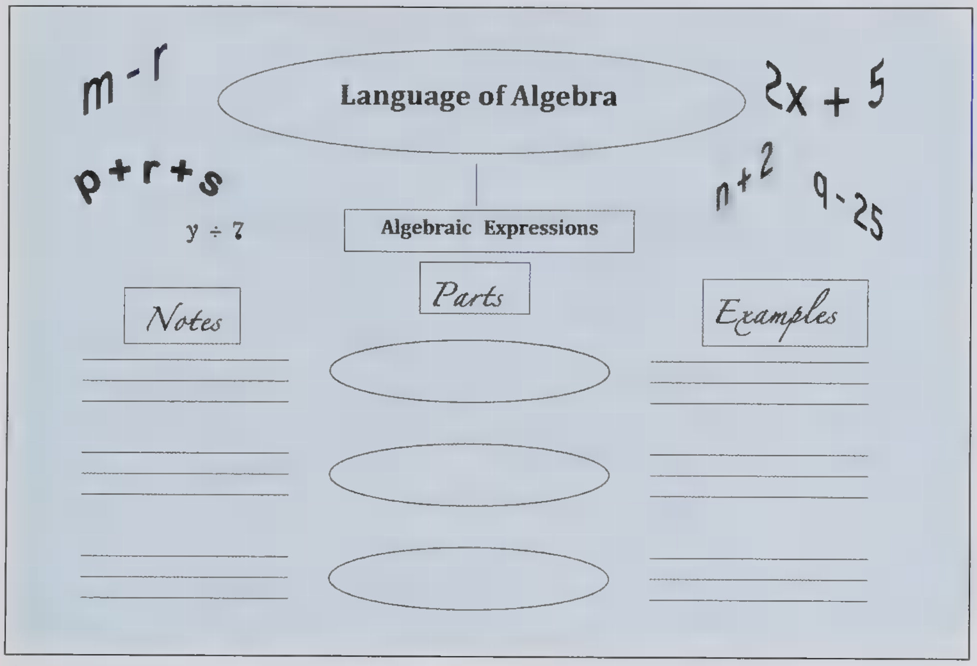


Fig. 2 *Variable*, *constant*, and *operation* would be appropriate entries in the ovals in the middle column.

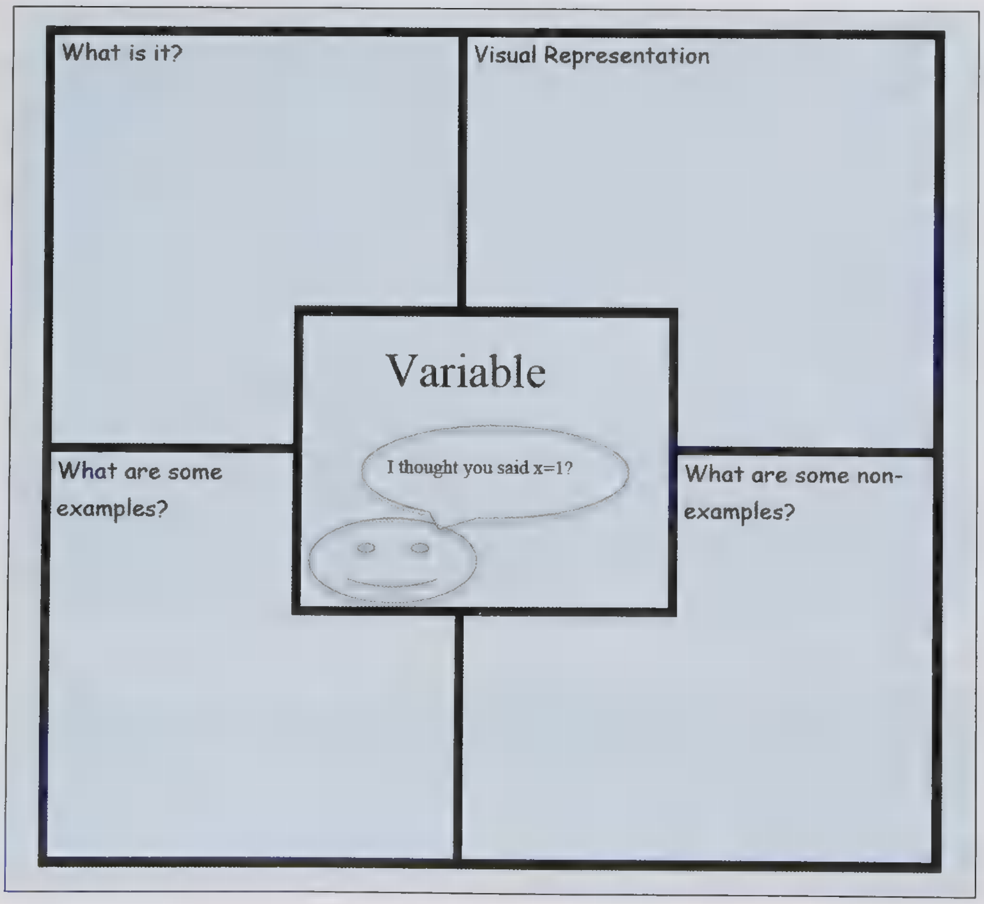


Fig. 3 A Frayer model is useful for some vocabulary.

Word Walls

Word walls are also useful within instructional units. A key idea is that word walls should be interactive, not static. After explicitly teaching words in the context of the unit, add definitions, examples, and diagrams to the words on the wall. Using nonexamples can help refine or clarify definitions (Adams 2003). In addition, real-life situations can provide context for algebra vocabulary and concepts (Paulsen 2007).

A helpful strategy is to start with informal definitions (while preteaching and assessing prior knowledge) and then transition to formal definitions (NCTM 2000). For example, the informal definition "a variable is a letter" may lead to "a variable is a symbol that represents a number" and finally to "a variable is a symbol, usually a letter, that is a quantity that can have different values." Informal definitions help students construct their own meaning, but formal definitions help them understand and apply concepts presented in mathematics textbooks (Adams 2003).

Ongoing interactive use of the word wall helps students see its value. As the year progresses, students use the word wall when answering verbal questions, when writing responses to essential questions on tests, and at other times when vocabulary usage is emphasized.

Graphic Organizers

Graphic organizers are beneficial within a unit of study to build and reinforce mathematics language. A graphic organizer entitled The Language of Algebra provides an opportunity to teach or reteach the parts of an algebraic expression by giving definitions and examples in the context of expressions (see **fig. 2**). In this specific organizer, the "parts" section (middle column) could list *variable*, *constant*, and *operation*, with notes and examples for each in the left and right columns. Similar language organizers could be developed for other topics.

A Frayer model is a specific graphic organizer that is useful when vocabulary terms are confusing or closely related (Barton and Heidema 2002). The model contains four sections: definition, facts, examples, and nonexamples (see **fig. 3** for an example related to the term *variable*). Both research (Adams 2003; Paulsen 2007) and personal experience demonstrate that nonexamples can be particularly powerful in helping refine and clarify definitions. When students ask, "How about this?" or "How about that?" they can refer to the example and nonexample sections. New misunderstandings that are uncovered

can be added to the "nonexample" section. Sometimes substituting sections to suit the situation can be useful—for instance, using essential characteristics and nonessential characteristics or symbolic representation and graphical representation as sections. Students frequently refer to their organizers during lessons or when reviewing for tests.

Provide Repetition and Support Long-Term Retention

All students benefit from repeated exposure to vocabulary; however, ELLs require more repetition to integrate vocabulary into their mathematical understanding. In addition, students may need assistance in organizing their vocabulary knowledge into long-term memory (Adams 2003). Using vocabulary words within context while referring to the definitions (Echevarria, Vogt, and Short 2010) can be helpful. Providing different examples or diagrams each time the word is used helps avoid confusion and brings depth to students' growing understanding.

Word Walls

Reinforcing vocabulary from the interactive word wall can support long-term retention. A simple idea is to take four to five minutes at the end of class to play password or charades, using words from current or previous word walls. Another idea is to encourage and facilitate instructional conversations (Cazden 2001) that can support long-term retention of mathematics language and build meaning about mathematical concepts (NCTM 2000). Word walls can scaffold these conversations. When small groups discuss a mathematics problem, points can be awarded for appropriate use of words from the word wall—for example, using words such as *formula*, *variables*, *equations*, *graphs*, and *order of operations* when discussing using algebra in the real world.

Graphic Organizers

The graphic organizers used throughout a unit can and should be revisited to support long-term retention. In addition, new graphic organizers can be introduced to review previously learned vocabulary and concepts. For example, an organizer with a formal definition, specific properties or special cases, and some examples could be used to review the concept of factors (see **fig. 4**).

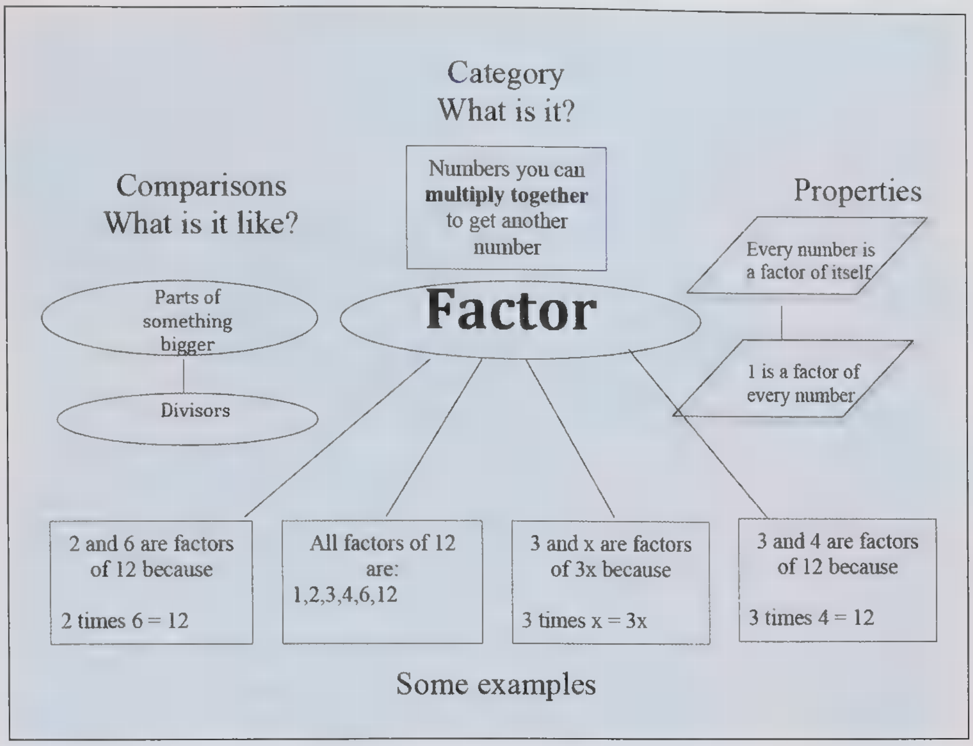


Fig. 4 This graphic organizer would be useful during review of the concept of factors.

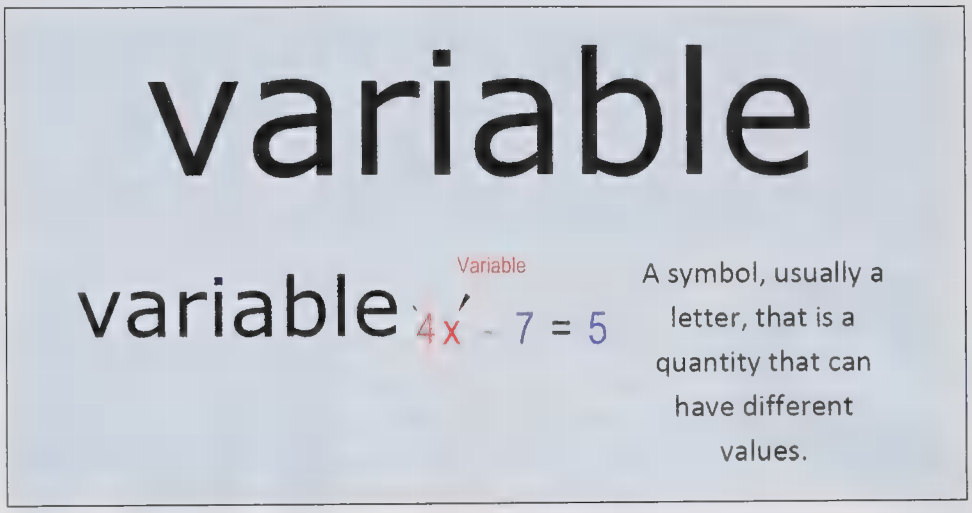


Fig. 5 This word wall entry can be folded so that only the vocabulary word is showing.

TEACHER AWARENESS

Along with reading research literature, mathematics teachers should build their own understanding of the challenges that their ELL students face. Awareness of the confusion caused by symbols and diagrams, concepts that can be represented with multiple terms, words that have multiple meanings, and the overlap between mathematics vocabulary and everyday usage can help teachers provide appropriate emphasis or explicit teaching.

HELPFUL HINTS

Word Walls

A simple way to make a word wall is to use a hanging pocket schedule organizer (typically used by elementary school teachers). After deciding on the unit vocabulary list (see **table 3**), type the words into a document (in landscape mode), with one word on each line. Center each word and enlarge it so that it fills a line of the paper. On the next line, type the word, its definition, a diagram, and an example. After printing, fold the paper so that the word is on one side and the expanded definition is on the other (see **fig. 5**). Slide the pieces into the organizer with the words showing. As the unit progresses and the words are discussed in context, reverse the paper so that the expanded definition is revealed.

Graphic Organizers

Many Internet sites—for example, CAST (www.cast.org) and Thinkport (www.thinkport.org)—have sample graphic organizers that can be used as is or customized. Teachers need not limit themselves to mathematics organizers; many excellent vocabulary organizers, such as Frayer models, come from other content areas. Providing a graphic organizer can help connect content within the unit and then can be used later as a review. Colored paper can assist with organization. In my class, colored paper means "keep it forever." Color makes important graphic organizers easy to find (I can say, "Pull out the red graphic organizer on variables"). At the end of the year, unit organizers make a good, concise way to review.

REFLECTIONS AND RECOMMENDATIONS

As I reflect on my experiences and those of my students, I am reminded of Jorge's confusion about mathematics vocabulary. His question has led me to increase my own awareness of the challenges related to mathematics vocabulary that ELL students face and strategies that I might use to support these students.

To help ELL students develop essential mathematical practices (CCSSI 2010), I recommend the use of word walls and graphic organizers to support vocabulary development. Specifically, I recommend the following:

- Select vocabulary words for a unit and post these on the day that the unit is introduced.
- Assess students' current understanding.
- Refer to the words throughout the unit, adding to the definitions and giving context.
- Provide frequent opportunities for students' misunderstanding to come to light.
- Use graphic organizers to help clarify the meaning of words and support long-term retention of vocabulary.

In addition to using word walls and graphic organizers, teachers should continue to investigate ideas available through books, journal articles, and websites (there are lots of good ideas out there). And, of course, listen to your students—that's the first step in supporting them.

REFERENCES

Adams, Thomasenia Lott. 2003. "Reading Mathematics: More Than Words Can Say." *The Reading Teacher* 56 (8): 786–95.

Barton, Mary Lee, and Clare Heidema. 2002. *Teaching Reading in Mathematics: A Supplement to Teaching Reading in the Content Areas: If Not Me, Then Who?* 2nd ed. Alexandria, VA: Association for Supervision and Curriculum Development.

Cazden, Courtney B. 2001. *Classroom Discourse: The Language of Teaching and Learning.* 2nd ed. Portsmouth, NH: Heinemann.

Common Core State Standards Initiative (CCSSI). 2010. Common Core State Standards for Mathematics. Washington, DC: National Governors Association Center for Best Practices and the Council of Chief State School Officers. http://www.corestandards.org/assets/CCSSI_Math%20Standards.pdf.

Echevarria, Jana, MaryEllen Vogt, and Deborah Short. 2010. *The SIOP® Model for Teaching Mathematics to English Learners.* Boston: Pearson.

Gee, James. 1996. *Social Linguistics and Literacies: Ideology in Discourses.* 3rd ed. London: Falmer.

Hall, Tracey, and Nicole Strangman. 2010. "Graphic Organizers." CAST (Center for Applied Special Technology). http://www.cast.org/publications/ncac/ncac_go.html.

Heinze, Kathryn. 2005. *The Language of Math.* St. Paul: Center for Second Language Teaching and Learning. http://kathrynheinze.efoliomn2.com.

Kotsopoulos, Donna. 2007. "Mathematics Discourse: 'It's Like Hearing a Foreign Language.'" *Mathematics Teacher* 101 (4): 301–5.

Moschkovich, Judit N. 1999. "Supporting the Participation of English Language Learners in Mathematical Discussions." *For the Learning of Mathematics* 19 (1): 11–19.

———. 2002. "A Situated and Sociocultural Perspective on Bilingual Mathematics Learners." *Mathematical Thinking and Learning* 4 (2 and 3): 189–212.

National Council of Teachers of Mathematics (NCTM). 2000. *Principles and Standards for School Mathematics.* Reston, VA: NCTM.

Paulsen, Kimberly. 2007. *The Iris Center: Case Study Unit: Algebra (Part 2): Applying Learning Strategies to Intermediate Algebra.* http://iris.peabody.vanderbilt.edu/case_studies/ICS-010.pdf.

Pimm, David. 1987. *Speaking Mathematically: Communication in Mathematics Classrooms.* London: Routledge and Kegan Paul.

Schlepegrell, Mary J. 2007. "The Linguistic Challenges of Mathematics Teaching and Learning: A Research Review." *Reading and Writing Quarterly* 23: 139–59.

NANCY S. ROBERTS, nancy.roberts@ct.gov, teaches mathematics at Howell Cheney Technical High School in Manchester, Connecticut. She is interested in exploring strategies to teach mathematics to all learners. **MARY P. TRUXAW,** mary.truxaw@uconn.edu, is an associate professor in the Neag School of Education at the University of Connecticut in Storrs. She is interested in how discourse can support mathematical understanding, especially in linguistically diverse mathematics classrooms.

LEARNING WITH CALCULATOR GAMES

Four graphing calculator games to entice your students to learn mathematics.

Bruce Frahm

Numerous studies show that students already spend a great deal of time playing electronic games. Why not capture that intense interest in gaming and involve students in learning by using a game format on a graphing calculator? Educational games provide a fun introduction to new material and a review of mathematical algorithms. Today's graphing calculators have a programming language that allows for interactive games and that can reinforce the mathematics taught in our classrooms. Specifically, games can be designed to assist students in developing mathematical skills as an incidental consequence of the game-playing process (Sedig 2008). Graphing and solving equations can be practiced in a game environment designed to entertain and involve students.

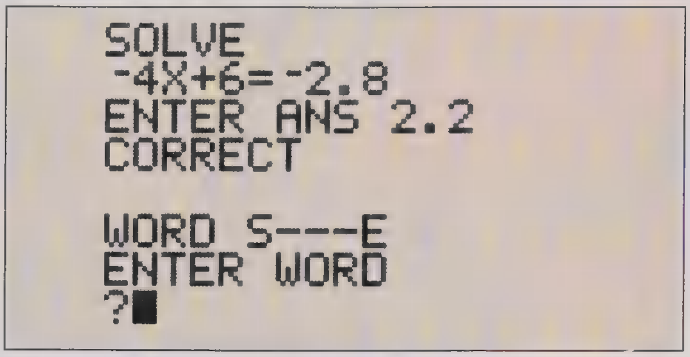


Fig. 1 Correctly solving an equation allows students to guess the mathematical vocabulary word.

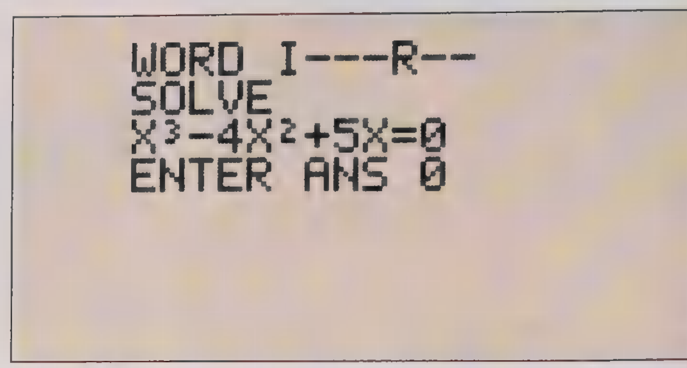


Fig. 2 When more than one solution exists, any correct answer suffices.

The games presented here have been designed and programmed by the author and are available for download into either Texas Instruments™ (TI-83, TI-84) or Casio® (fx-9750G, fx-9860G, fx-CG10) graphing calculators. Each game has been thoroughly field-tested in several classrooms; student input has been a valuable component of the design process. I have used these programs to introduce a topic, review for an assessment, and stimulate students' interest in learning more mathematics. An added benefit to using games to teach is being able to reinforce mathematical thinking in an entertaining format, engaging students who have a limited interest in mathematics and enticing them to learn abstract mathematical principles.

Both NCTM's Standards (2000) and the Common Core State Standards (CCSSI 2010) address building on earlier experiences and student interaction to deepen an understanding of mathematics. Any attempt to involve students in actively solving and graphing equations develops the kind of expertise mentioned in the CCSS mathematical practices. Using games designed to encourage practice will create the proficient students mentioned in Mathematical Standard 1 by reinforcing the processes used in creating a correct response (CCSSI 2010, p. 6). The content standard from CCSS Functions-Building Functions 3, which encourages students to experiment with the parameters of graphing (CCSSI 2010, p. 70), matches a game format.

The programs presented here are adaptations of board games or television shows that students may be familiar with, making the experience more memorable for them.

WORD WHIZ

Word Whiz plays the word game that most resembles the toss-up words on the television show *Wheel of Fortune*®. This program, called **WORDWHIZ**, is available only on the TI calculator. The game can be used in the classroom in several ways. The calculator could be placed under a document camera and the image projected to the entire class, so that the game could be used as a review for solving equations. Or, if each student has a calculator, each can be challenged to achieve a high score working independently.

As students correctly solve an equation from second-year algebra (see **fig. 1**), the calculator randomly picks a letter to display; after the letter appears, students try to guess the mystery word. If the word is correct, they will then proceed to the next word and more equations to solve; incorrect guesses allow students to solve another equation to reveal more letters to help them discover the mystery word.

The program was intended as an end-of-year review for intermediate algebra or for a beginning exercise in precalculus and incorporates the vocabulary words found in most textbooks. Teachers can alter the words used in the program by changing the code; very little programming knowledge is needed to replace the words to fit another curriculum. The toss-ups begin as five-letter words and increase by one letter in each round as the equations increase in difficulty. The program was designed for five rounds, with each round allowing the student six chances to solve an equation and guess the word. Each equation solved incorrectly subtracts 5 points from the initial score of 100. For words of length greater than five letters, there is a penalty of 5 points for needing more than five letters to decipher the word.

The calculator can offer multistep linear, quadratic, exponential, logarithm, and radical equations. I ask students to hand in their attempts to solve each equation, along with the words that they found in the toss-ups. Equations that produce multiple answers will accept any correct answer for the next letter to appear in the toss-up (see **fig. 2**). Guessing words by identifying the letters also applies to our school literacy initiative. This game takes about twenty minutes to play, depending on students' recognition of the vocabulary words.

TAKE IT OR LEAVE IT

This calculator game simulates the television game show *Deal or No Deal*®, in which contestants try to identify a briefcase with 1 million dollars in it. The calculator will randomly place twenty-six amounts

SASHKINW/ISTOCKPHOTO

Deal or No Deal Suitcases			
$0.01	$100	$5,000	$300,000
$1	$200	$10,000	$400,000
$5	$300	$25,000	$500,000
$10	$400	$50,000	$750,000
$25	$500	$75,000	$1,000,000
$50	$750	$100,000	
$75	$1,000	$200,000	

Fig. 3 Instead of in suitcases, dollar amounts are stored in one of the stat lists.

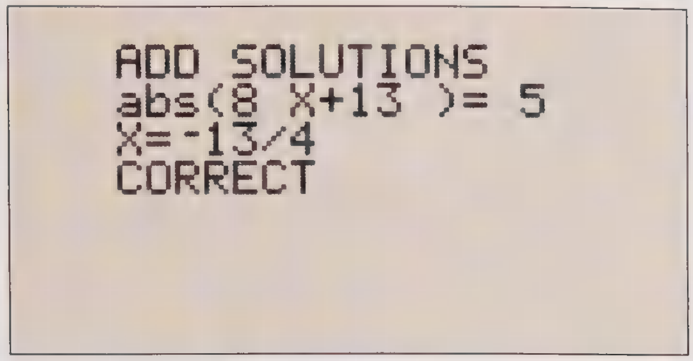


Fig. 4 When prompted to do so, students need to sum both solutions to move forward in this game.

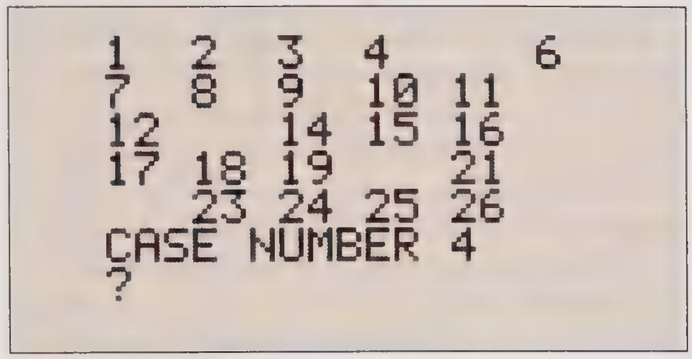


Fig. 5 The available cases are always displayed.

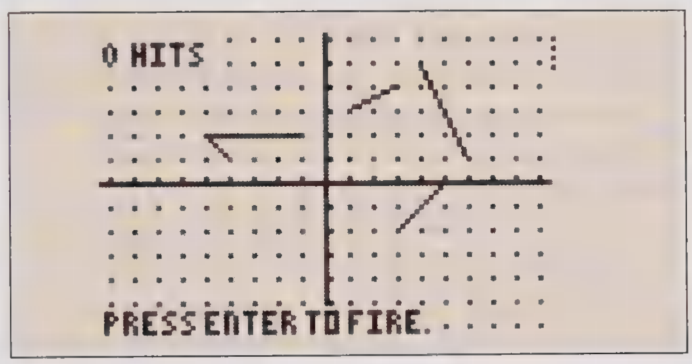


Fig. 6 Five ships are placed on the coordinate plane.

of money (see **fig. 3**) into the cases, represented by the numbers 1–26. After correctly solving an equation, students are then able to pick a case that may hold the 1 million dollars. To find out what is in their case, students must open the other twenty-five cases or sell their case to "The Calculator." Students may stop opening cases at any level, end the game, and accept The Calculator's offer for their case. They may decide to continue opening all the other cases until the one case that they chose initially is left. The Calculator tries to buy the student's case at the end of each level by offering a percentage of the average amounts in the unopened cases. The first several levels have a low percentage of money so students are challenged to improve their deal by solving more equations. Each time the program is run, the dollar amounts in each briefcase will change, and the equations are randomly generated to be completely different.

Equations vary from one to two steps, with a variable on both sides of the equals sign. Difficulty increases with each level. Quadratic, absolute value, and logarithm equations may have more than one answer, thus requiring students to add the correct solutions together for the program to recognize the correct response (see **fig. 4**). The intent of this game is to practice solving equations, not to teach students problem-solving methods. Incorrect answers allow students a chance to reenter a solution but subtracts $500 from The Calculator's offer. The numbers of the unopened cases are always displayed on the screen to prompt students not to request cases that have already been opened (see **fig. 5**). Students are required to turn in all the problems and

amounts offered by The Calculator at the end of the period (see **activity sheet 1**; customizable activity sheets for all activities are available at **www.nctm.org/mt**).

Students enjoy the competition with one another and will often rerun the program to improve their chances of winning more money. Competition of this type establishes a great review: The deals are different each time the program is run, and the equations are randomly generated. Most students take an entire period to ensure getting the best dollar amount possible. I keep a leader list on the board to encourage students who quit early with a lower dollar amount to try again.

LINEAR BATTLESHIP

This calculator program simulates the familiar board game Battleship® (by Hasbro™), in which players try to sink one another's ships by identifying coordinates on a grid. The TI program is BATTLESP, and the Casio program is BAT.SHIP. In this adaptation of the board game, students try to sink the calculator's ships given fifteen chances

```
ENTER TORPEDO  1
Y=X-5
ENTER TORPEDO  2
Y=-X-3
ENTER TORPEDO  3
Y=2
ENTER TORPEDO  4
Y=
```

Fig. 7 This student will have hits with her first three torpedoes.

```
THE VIEW TIME
 WILL INCREASE
  WITH EACH TRY.
CHOOSE:
1 QUADRATIC OR
2 ABSOLUTE VALUE
PRESS 1 OR 2
```

Fig. 8 Students can select the version of the game that they would like to play.

to graph lines (torpedoes) in slope-intercept form (see **fig. 6**). The ships are line segments of different lengths and slopes that the program places at random. Lines that intersect the ship do not count as hits; only lines that match the exact slope and y-intercept count as a hit. The length of each ship varies to represent various types of ships (e.g., destroyer, aircraft carrier, submarine). The location of each ship is randomly selected; therefore, sometimes the ships may touch one another. The screen's dimensions are large enough that students can try different slopes and y-intercepts to reinforce the concept of matching an equation to the actual graph of the line. Each incorrect line (torpedo) deducts 5 points from the base score of 100 (see **fig. 7**).

The grid on the screen allows students to count vertical and horizontal change for the ship's slope; however, when the y-intercept is off the screen, students have a further challenge. Each line (torpedo) is left on the screen to assist students in altering their shot. I provide students with a blank grid and require them to record each linear equation

MATTY2X4/ISTOCKPHOTO

and sketch the corresponding graphs (see **activity sheet 2**). When all five ships are sunk, the program gives a final score that assesses the student's ability to determine the equation of a line.

I use a classroom set of calculators and have a competition in class to see who can get the highest score. Because students each have a different arrangement of ships, they often will retry the game to improve their skills at graphing and ultimately improve their score. I have also used the score as an assessment; students must verify their score by showing the display screen of the calculator. Students always enjoy the opportunity to improve their score on an assessment. Once a student masters the concept of slope and y-intercept, they are quick to score a perfect 100.

NAME THAT GRAPH

In the 1980s television game show *Name That Tune®*, contestants were given a chance to win money by identifying a song after hearing only several notes. The calculator game "Name That Graph" follows the same pattern, using a variety of graphs instead of songs. Students try to determine the name—that is, the equation—of a graph in a limited time. The strategy of identifying an equation from the graph requires deciphering changes in the basic appearance of the *parent graph*, defined as "a family of graphs with the parent being the simplest" (*Prentice Hall Algebra 2* [2011], p. 100). When students identify appropriate points and follow the trends in the equations, they begin to learn the shape of the graphs. Transformations, reflections, and dilations can make the graphs more interesting but also may create difficulty for students in interpreting the equations. The NCTM Standards, too, emphasize the importance of understanding not only how different equations create different graphs but also how changing the parameters in an equation can alter it to "look" different (NCTM 2000, p. 297).

"Name That Graph" (NTGQUAD and NTGQUAD2) uses transformations that can be applied to absolute value $y = a|(x - h)| + k$ and quadratic graphs $y = a(x - h)^2 + k$ (see **fig. 8**) to identify horizontal shifts (h), vertical shifts (k), and dilations or

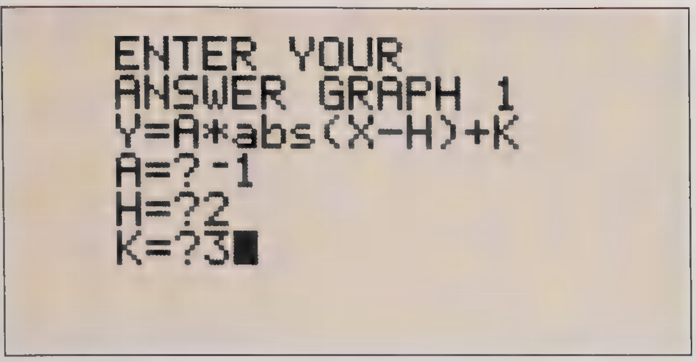


Fig. 9 Students have only about one second to view the graph before entering a guess.

reflections (a) (see **fig. 9**). The graph appears in a timed window (for about one second) and students then have a chance to enter their choices for a, h, and k. In the brief time frame, students are to visualize the horizontal and vertical shifts and then determine the value of the dilatation or reflection. Each incorrect answer subtracts 2 from the base score of 100 and allows more time to see the graph. After ten graphs, the total score is displayed. Two versions of the game are available. One is based on predetermined values for each parameter, giving teachers the option to change the code and either increase or decrease the difficulty level. The second version uses random values within reasonable constraints of the display screen, thus making the program available to play multiple times.

The game allows students to experiment with positive and negative values of the parameter to see how the graph can shift, stretch, or reflect. Students complete **activity sheet 3** and hand this in. Students become adept at identifying the vertex to represent the graph but sometimes have difficulty with the dilation. I offer help with determining the dilation by showing them the only possible values that the program will use for a: $\pm 1/2$, ± 1, ± 2. I use this game frequently for review, but it also can be used to introduce graphing with transformations.

To play any of these games, students may need to be shown how the graphing calculator accepts answers by pressing the ENTER or the EXE button. Syntax errors will cause the programs to crash and will require students to rerun the program from the beginning, resetting their score. The negative key (-) must be used instead of the subtraction key. Fractions on the TI calculator use the division key. The Casio calculator has a fraction key, but the division key can be used as well. Fractions need not be reduced for the calculator to accept the answer as correct, but students should be instructed to reduce all fractions for practice. Students must avoid pressing the ON key; the program will break and could allow the student to alter the code. All the programs can be downloaded to a computer and then transferred to calculators using programs provided by each calculator manufacturer.

DOING MATHEMATICS WHILE PLAYING

Playing games to teach mathematics is not a new concept. However, the graphing calculator gives students an opportunity to use current handheld technology and learn mathematics as a side effect of playing a game. Technology is changing almost daily, with color and high-resolution graphics, but the programming language used by most calculator companies remains similar to BASIC, allowing game programs to be easily developed. These programs involve an element of discovery in a game format that provides students the incentive to practice mathematical concepts in an entertaining and engaging fashion.When used as a formative assessment, these games encourage students to play repetitively to improve their score and demonstrate mastery.

Students who seem reluctant to get involved in classwork or homework enjoy the opportunity to do mathematics while playing a game. My students look forward to game days, not realizing that they are learning and demonstrating their proficiency with mathematics.

BIBLIOGRAPHY

Algebra 2. 2011. Upper Saddle River, NJ: Pearson/Prentice Hall.

Common Core State Standards Initiative (CCSSI). 2010. Common Core State Standards for Mathematics. Washington, DC: National Governors Association Center for Best Practices and the Council of Chief State School Officers. http://www.corestandards.org/assets/CCSSI_Math%20Standards.pdf.

National Council of Teachers of Mathematics (NCTM). 2000. *Principles and Standards for School Mathematics*. Reston, VA: NCTM.

Sedig, Kamran. 2008. "From Play to Thoughtful Learning: A Design Strategy to Engage Children with Mathematical Representations." *The Journal of Computers in Mathematics and Science Teaching* 27 (1): 65–101.

BRUCE FRAHM, frahm.bruce@brevardschools.org, teaches grades 9-12 at Eau Gallie High School, Melbourne, Florida. He is interested in incorporating technology with games and puzzles into his mathematics classroom.

more4U

Codes for the programs for these games as well as customizable activity sheets are available at **www.nctm.org/mt**.

AUGUST

NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

With the new school year, the Calendar introduces a new feature: the Problem of the Month. The Problem of the Month for August is problem 28. Student solutions submitted to Séan Madden, smadden@greeleyschools.org, and Ricardo Diaz, Ricardo.diaz@unco.edu, may be published in a future issue of the journal.

1

Wally's Wheels bike shop offers a deal for two bikes: Buy one at full price, pay half price for the second. All bikes at Wally's are \$130. At Bobby's Bicycle Boutique, customers pay full price, \$95, for the first bike; they then receive 20% off the second bike and 30% off the third. Which store offers the better deal for two bikes?

2

An Olympic gymnast can earn an average score of 16.5 points for each of two vaults. Another can earn an average score of 15.5 points. Suppose that the first gymnast falters, losing 2.375 points on her first vault and 1.575 on her second vault. What is the average score that the second gymnast must earn to win the gold?

3

You have six canisters of the same size and shape. Five of these canisters contain exactly the same amount of gold, but the sixth one contains slightly more than the rest. If you are allowed to use a balance scale only twice, is it possible to determine the canister with the most gold? If yes, explain your procedure.

4

A city bus travels its usual route. At the first stop, 1 person boards; the total number of passengers is now divisible by 5. At the second stop, 2 people get off; the total number of passengers is now divisible by 2. At the third stop, 6 people board; the total number of passengers is now divisible by 4. If the bus holds at most 25 riders, how many passengers were on the bus before the first stop?

5

A high school class consists of 6 seniors, 10 juniors, 12 sophomores, and 4 freshmen. Exactly 2 of the juniors are female, and exactly 2 of the sophomores are males. If two students are randomly selected from this class, what is the probability that the pair consists of a male junior and a female sophomore?

6

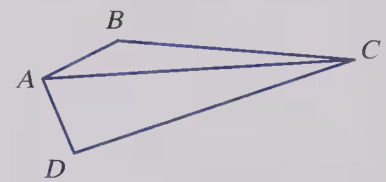
By factoring each term, find a rule for the following sequence:

16, 24, 40, 56, 88

What are the next two terms?

7

Quadrilateral $ABCD$ has $AB = 5$, $BC = 8$, $CD = 15$, and $DA = 4$. If diagonal AC has integral length, find AC.



8

John has \$170 to spend on a party. At the store he sees that he can buy 14 bottles of specialty soda and 6 large pizzas or 2 bottles of specialty soda and 13 large pizzas for exactly \$170. How much does each soda and each large pizza cost?

9

The pitcher's mound in a Major League Baseball diamond is not in the center of the diamond; it is equidistant from first and third bases and 60 ft. 6 in. from home plate. If the four bases of the diamond (square) are 90 ft. apart, how far is the pitcher's mound from the center of the diamond? Give your answer in feet and inches to the nearest inch.

10

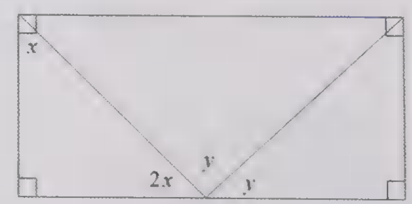
Let $GCD(a, b)$ represent the greatest common divisor of a and b, and let $LCM(a, b)$ represent the least common multiple of a and b. Use $GCD(a, b)$, $LCM(a, b)$, and the product $GCD(a, b) \cdot LCM(a, b)$ for each of the following pairs

$(4, 6), (5, 9), (14, 35), (30, 45)$

to make an observation about the products.

11

Solve for x and y in the figure:



A school bus has 15 rows with two benches each; each bench seats three. Sara prefers to sit directly next to Megan. Tomas prefers to sit next to Sara, possibly across the aisle from her. Megan has no preferences. No child sits in the same seat twice. If the children ride the bus to school and back every day, how many days will pass before they break one of their rules?

How many nonoverlapping right isosceles triangles with hypotenuse length 2 can fit inside a square of area 8?

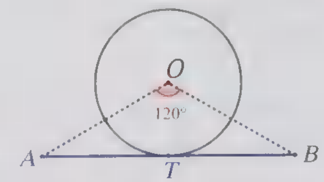
George, Laura, and Jane are running a 2-mile race. George runs twice as fast as Laura. Running at a constant pace, Laura will finish the race 10 min. after George. If Laura had a 5-min. head start, she and Jane would finish at the same time. How long a lead can George give Laura and Jane so that he does not lose?

The sum of three positive integers is 30. The smallest integer, x, is a factor of 30. The second integer is 1 less than twice the quotient of 30 and x, and the third integer is 3 more than the second integer. Find the three integers.

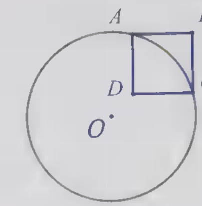
16 Ada and Bea take turns rolling a pair of dice. Ada always bets \$1 that the sum of the dice will be 7. Bea also places a \$1 bet, but she always bets on a pair of possible sums; for example, she may bet on a sum of 3 or 5. If Ada and Bea break even in the long run, how many different bets can Bea place?

17 Find the area of the triangle with vertices at A (–8, –6), B (6, 9), and C (8, –14).

18 Segment AB with midpoint T is tangent to circle O at T. If $m\angle AOB = 120°$ and tangent AB has length 6 ft., what is the area of circle O?

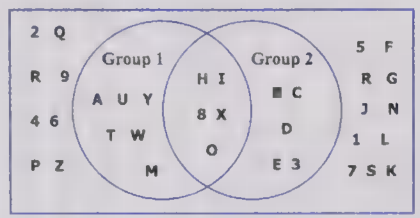


19 Vertices A and C of square $ABCD$ lie on circle O. The radius of the circle is 10 in. If the area of $ABCD$ is 50 in.^{2}, what is the area of the sector of circle O that is subtended by arc AC?



20 After baking a rectangular cake with length 10 in., width 8 in., and height 4 in., you add a 1/3-in. layer of icing to all the exposed sides of the cake. What is the total volume of the finished cake?

21 Examine the Venn diagram below. What defines the criteria for inclusion in group 1 and group 2? Where does the letter V belong?



22 Sam lives 4 blocks north and 7 blocks west of John in a town where streets intersect at right angles. To get to John's house, Sam can ride his bike (on the streets) at a constant rate of 18 mph, or he can cut through the neighbors' yards (a direct route), walking at 4 mph. Which mode of transportation will allow Sam to reach John's house faster? (1 block = 0.1 mile)

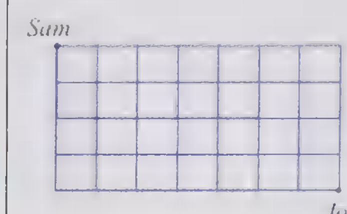
23 Consider again the problem for August 22. Suppose that Sam decides to bike but, after traveling 4 blocks, gets a flat tire and has to walk the rest of the trip. Would he have been better off walking the direct route through the neighbors' yards?

24 Line l passes through points (0, 1) and (6, 5). What is the y-intercept of line k if k is perpendicular to l and passes through the point (2, 4)?

25 In basketball, a "suicide" drill is sprinting the distance from the baseline to the nearer foul line and back, then to the half-court line and back, then to the farther foul line and back, and finally to the opposite baseline and back. Find the total distance sprinted if the farther foul line is 63 ft. from the starting baseline. (Each foul line is midway between a baseline and the half-court line.)

26 A cistern is filled through five canals. With only the first canal open, the cistern fills in 1/3 day; with only the second open, it fills in 1 day; with only the third open, in 2 1/2 days; with only the fourth open, in 3 days; and with only the fifth open, in 5 days. If all the canals are open, how long will it take to fill the cistern?

27 Consider again the problem for August 22. Since the streets intersect at right angles, they form a rectangular grid like the one shown. If Sam bikes only east or south on the streets from his house to John's, how many different paths can he choose from?



28 The sum of all integers from 1 to n, inclusive, is equal to the sum of all integers from n to 49, inclusive. What is the value of n?

29 A student rolls two 6-sided red dice and one 6-sided blue die. What is the probability that the number showing on the blue die exceeds both the numbers on the red dice?

30 Four beads will be strung on a chain to make a necklace. The clasp is tiny, so the beads can slide over it easily. Thus, the two necklaces shown represent the same arrangement of beads on the chain:

If each bead can be red or black, in how many different ways can the beads be arranged on the chain?

31 How many times in a 24-hour day are the minute hand and the hour hand perpendicular to each other?

SOLUTIONS to calendar

Looking for more Calendar problems?

Visit www.nctm.org/publications/calendar/default.aspx?journal_id=2 for a collection of previously published problems—sortable by topic—from *Mathematics Teacher.*

Edited by ***Margaret Coffey,*** *Margaret .Coffey@fcps.edu, Thomas Jefferson High School for Science and Technology, Alexandria, VA 22312, and* ***Art Kalish,*** *artkalish@ verizon.net, Syosset High School (retired), Syosset, NY 11791*

Calendar Problems 1–25 were created by the Mathematical Problem Solving class (Spring 2012) at Saint Joseph's University in Philadelphia, under the direction of Agnes M. Rash, professor of mathematics. Class members included Joshua Bargiband, Sarah Bell, Matthew Black, Moira Devlin, Matthew Emery, Shane Flannery, Michael Kehoe, Erica Latz, Elisa Miller, and Reesa Roccapriore. Special thanks to Moira Devlin and Elisa Miller for editing many of the problems to fit into the allocated space and to Joshua Bargiband for creating more than the number of problems for which he was responsible.

Problems 28, 29, 30, and 31 were submitted by Charles Kicey. He and co-author John Seppala wrote the problems and solutions for the high school contests hosted by Valdosta State University in Georgia, where they teach.

The Editorial Panel of *Mathematics Teacher* is considering sets of problems submitted by individuals, classes of prospective teachers, and mathematics clubs for publication in the monthly Calendar. Send problems to the Calendar editors. Remember to include a complete solution for each problem submitted.

Other sources of problems in calendar form available from NCTM include *Calendar Problems from the "Mathematics Teacher"* (a book featuring more than 400 problems, organized by topic; stock number 12509, \$22.95) and the 100 Problem Poster (stock number 13207, \$9.00). Individual members receive a 20 percent discount off this price. A catalog of educational materials is available at www.nctm.org.–Eds.

1. Bobby's Bicycle Boutique. Two bikes at Wally's Wheels cost \$130 + 0.5(\$130) = \$195. At Bobby's, a customer pays \$95 + 0.8(\$95) = \$171. A customer saves \$24 by purchasing two bikes at Bobby's.

2. More than 14.525. The first gymnast's total score is $2(16.5) - 2.375 - 1.575 = 29.05$. The second gymnast must earn an average score greater than $29.05/2 = 14.525$ to win the gold.

3. Yes. There are two possibilities. Call the canister with the greater amount of gold *X*. Place three canisters on each side of the scale; whichever side weighs more contains *X*. Take two of the three canisters from the heavier side and place one on each side of the scale. If one is heavier than the other, you have found *X*; if both sides weigh the same, the third canister is *X*.

Alternate solution: Separate the canisters into groups of two and place one group on each side. If one side is heavier, one of those two canisters is *X*, and one additional weighing will determine which it is. If the two sides weigh the same, then *X* is one of the two canisters in the third pair. One additional weighing will determine which it is.

4. 19. Let p = the number of passengers on the bus before the first stop. Then $p + 1$ is divisible by 5; $p - 1$ is divisible by 2; and $p + 5$ is divisible by 4. If $p - 1$ is divisible by 2, then $p + 1$ is also divisible by 2. If a number is divisible by 2 and by 5, it is divisible by 10, so $p + 1$ is a multiple of 10; therefore, p must be 9 or 19. (The bus holds at most 25 riders.) We have $p + 5 = 14$ or 24; only 24 is divisible by 4, so $p = 19$.

5. 5/31. There are 32 students in the class. Since 2 of the juniors are female, 8 juniors are male. Since 2 of the sophomores are male, 10 of the sophomores are female. We find the probability of selecting a male junior and a female sophomore in either order:

$$\frac{8}{32}\cdot\frac{10}{31}+\frac{10}{32}\cdot\frac{8}{31}=\frac{5}{31}\approx .161$$

6. 104, 136. Note that $16 = 2^3 \cdot 2$, $24 = 2^3 \cdot 3$, $40 = 2^3 \cdot 5$, $56 = 2^3 \cdot 7$, and $88 = 2^3 \cdot 11$. The rule is multiply 2^3 by consecutive primes. We have $2^3 \cdot 13 = 104$ and $2^3 \cdot 17 = 136$.

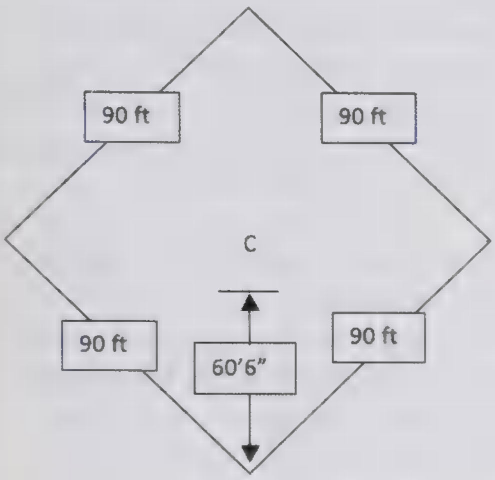
7. 12. Apply the triangle inequality theorem to triangles *ABC* and *ADC*. In $\triangle ABC$, $3 < AC < 13$. In $\triangle ADC$, $11 < AC < 19$. Thus, $11 < AC < 13$. If *AC* is an integer, then $AC = 12$.

8. \$7 for a bottle of soda, \$12 for a pizza. Let s = the cost of a soda and let p = the cost of a large pizza. Then $14s + 6p = 170$ and $2s + 13p = 170$. To solve the system, multiply the second equation by 7 and subtract the first equation from the second to obtain $85p = 1020 \rightarrow p = 12$, $s = 7$.

Alternate solution: Write the system of equations as before. Since 170 is the constant in both equations, we have $14s + 6p = 2s + 13p \rightarrow s = (7/12)p$. Substituting, we have $2(7/12)p + 13p = 170 \rightarrow 14p + 156p = 12 \cdot 170 \rightarrow p = 12$, $s = 7$.

9. 3 ft. 2 in. The pitcher's mound lies between second base and home plate, a distance that is the hypotenuse of an

isosceles right triangle with legs 90 ft. Therefore, the distance from second base to home plate is $90\sqrt{2} \approx 127.279$ ft., and the distance from the center to home plate is approximately 63.640 ft. The pitcher's mound is $63.640 - 60.5 = 3.140$ ft. from the center, or (to the nearest in.) 3 ft. 2 in.



10. $GCD(a, b) \cdot LCM(a, b) = ab$. We present the requested values in the table below.

11. $x = 30°$, $y = 60°$. The acute angles of a right triangle sum to 90°, so $x + 2x = 90° \rightarrow x = 30°$. We also have $2x + y + y = 180° \rightarrow y = 60°$.

12. 22 1/2 days. Create a diagram that shows the possible arrangements for the children in a single row. Let S stand for Sara, let T stand for Tomas, let M stand for Megan, and let X stand for a vacant seat on a bench. Three possible arrangements follow:

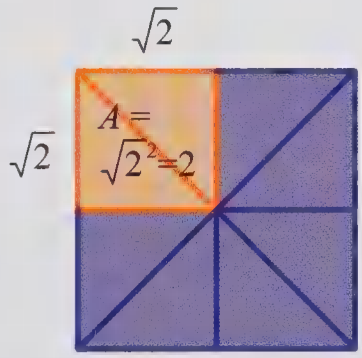
MST XXX   *XMS TXX*   *XXX MST*

Other arrangements are possible—for example, *TSM XXX*—but this arrangement could replace the first one listed above; it does not provide an additional arrangement because Sara sits in the middle seat in both cases, thus violating a condition of the problem. We find 3 possible arrangements per row times 15 rows for a total of 45 trips. The trip home on the 23rd day will force the children to break one of their rules.

Solution 16

Sum	2	3	4	5	6	7	8	9	10	11	12
***P*(sum)**	1/36	2/36	3/36	4/36	5/36	6/36	5/36	4/36	3/36	2/36	1/36

13. 8. A right isosceles triangle with hypotenuse 2 has legs of length $\sqrt{2}$. A pair of these triangles that share the same hypotenuse form a square of area $(\sqrt{2})^2 = 2$. Four such pairs arranged as shown in the figure entirely fill a square of area 8. Because the entire area of the square is covered, there is no better solution.



14. 5 min. at most. Since George runs twice as fast as Laura, she runs 1 mile while George runs 2 miles, which is the length of the race. Since Laura will finish 10 min. after George, she runs 1 mile in 10 min. George's rate is 1 mile in 5 min. To find Jane's rate, observe first that Laura will require 20 min. to run the race. Jane will require 15 min. to run the same distance; her rate is 7.5 min. per mile. Thus, George's rate is 2.5 min. per mile faster than his nearest competitor (Jane). The length of the race is 2 miles, so George can give the others at most a 5-min. lead.

15. 5, 11, 14. The smallest integer is x. Express the second integer as $2(30/x) - 1$ and express the third integer as $2(30/x) - 1 + 3 = 2(30/x) + 2$. We have $x + 2(30/x) - 1 + 2(30/x) + 2 = 30$. Combine like terms: $x + 120/x - 29 = 0$. Multiply both sides by x and then factor: $x^2 - 29x + 120 = 0 \rightarrow (x - 5)(x - 24) = 0 \rightarrow x = 5$. We reject the second solution because x must be a factor of 30. The second integer is $2(30/5) - 1 = 11$, and the third integer is $2(30/5) + 2 = 14$. Confirm that $5 + 11 + 14 = 30$.

16. 9 bets. If Ada and Bea break even in the long run, we will assume that Bea places her bets on pairs of sums that are as likely to occur as the sum of 7. The probability distribution for the sum of two dice shown in the table above may be helpful.

Consider the underlying sums:

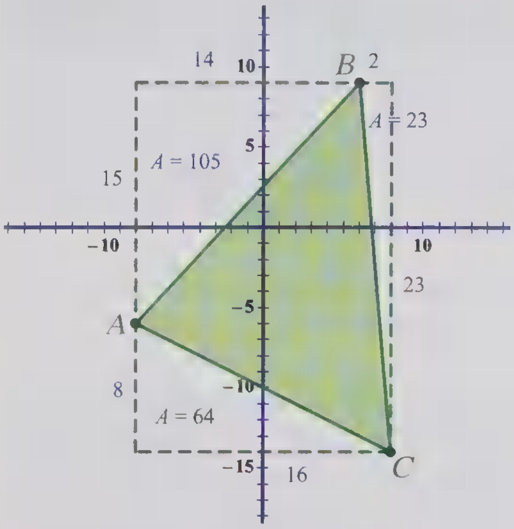
$$\frac{6}{36} = \frac{1}{36} + \frac{5}{36}, \quad \frac{6}{36} = \frac{2}{36} + \frac{4}{36}, \quad \frac{6}{36} = \frac{3}{36} + \frac{3}{36}$$

The symmetric probability distribution indicates that $P(2) = P(12) = 1/36$ and $P(6) = P(8) = 5/36$, so there are four different probability statements that correspond to the first sum: $P(7) = P(2 \text{ or } 6)$; $P(7) = P(2 \text{ or } 8)$; $P(7) = P(6 \text{ or } 12)$; and $P(7) = P(8 \text{ or } 12)$. Find four different statements that correspond to the second sum and only one that corresponds to the third sum, for a total of 9 different bets.

17. 176 units^2. Plot the points A, B, and C. Through these points, construct lines parallel to the x- and y-axes to form a rectangle (see fig. on p. 44). The area of $\triangle ABC$ can be found by subtracting the areas of three right triangles from the area of the rectangle, which measures $16 \times 23 = 368$ units^2. We have $368 - 105 - 23 - 64 = 176$ units^2.

Solution 10

a	*b*	***GCD*(*a*, *b*)**	***LCM*(*a*, *b*)**	***GCD*(*a*, *b*) · *LCM*(*a*, *b*)**
4	6	2	12	24
5	9	1	45	45
14	35	7	70	490
30	45	15	90	1350



(The department editors resist the temptation to offer a list of alternate approaches, believing that students will enjoy the challenge of seeing how many different solutions they can find.)

18. 3π ft.^{2}. Segment OT, the radius to the point of tangency, is perpendicular to tangent AB. Since T is the midpoint of $\overline{AB}$, $\triangle AOT$ and $\triangle BOT$ are congruent right triangles. They are 30-60-90° right triangles, with 60° angles at O. Since $AT = 6/2 = 3$, $OT = \sqrt{3}$. So the area of the circle $A = \pi(\sqrt{3})^2 = 3\pi$ ft.^{2}.

19. $50\pi/3$ in^2. Since the area of the square is 50, the length of a side of the square is $\sqrt{50}$. So the diagonal of the square is $\sqrt{50} \cdot \sqrt{2} = \sqrt{100} = 10$. Since $\triangle AOC$ is equilateral, we know that the angle formed by radii OA and OC is 60°. Thus, the area of the sector subtended by arc AC is $\pi \cdot 10^2/6 = 50\pi/3$ in.^{2}.

20. Very close to 400 in.^{3}. The cake's new dimensions are 10 2/3 × 8 2/3 in. × 4 1/3 in. We have $(32/3)(26/3)(13/3) = 10{,}816/27 \approx 400.59 \approx 400$ in.^{3}.

21. Symmetry; group 1. Each element in group 1 contains a vertical line of symmetry. Each element in group 2 has a horizontal line of symmetry. Elements in the intersection of groups 1 and 2 have both a horizontal and a vertical line of symmetry. Therefore, V belongs in group 1 but not in the intersection of groups 1 and 2.

22. Biking. The biking distance is 11 blocks = 1.1 mi. The walking distance can be found from the Pythagorean theorem: $d = \sqrt{4^2 + 7^2} = \sqrt{65} \approx 8.06$ blocks = 0.806 mi. Sam can bike 1.1 miles in 1.1/18 hr. ≈ 0.0611 hr., or 3 min. 40 sec. Sam can walk 0.806 miles in 0.806/4 hr. ≈ 0.2015 hr., or 12.1 min. Biking is faster. If we do not care to know how much faster biking is, then the calculations are unnecessary. Sam bikes at a rate that is 4.5 times faster than he walks. Therefore, walking is the better choice only when the ratio

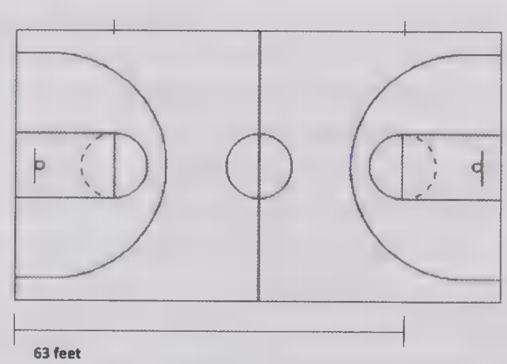
$$\frac{\text{biking distance}}{\text{walking distance}}$$

is greater than 4.5. In this problem, the ratio is roughly 11/8, far less than 4.5.

23. No. Biking the first four blocks, or 0.4 mi., required 0.4/18 hr. = $0.0\overline{2}$ hr. = 1 min. 20 sec. Walking the remaining 0.7 mi. required 0.7/4 hr. = 0.175 hr. = 10.5 min., for a total travel time of 11 min. 50 sec., still less than the 12.1 min. to walk directly to John's house.

24. 7. The slope of l is $(5 - 1)/(6 - 0) = 2/3$, so k has slope $-3/2$. An equation for k is thus $y - 4 = (-3/2)(x - 2) \rightarrow y = 7$ when $x = 0$.

25. 420 ft. The foul lines and the half-court line divide the entire basketball court into quarters. We know that 63 ft. is 3/4 the length of the court, so 21 ft. is 1/4 the length of the court. We sum the four portions of the sprint: $2(21 + 42 + 63 + 84) = 420$ ft.



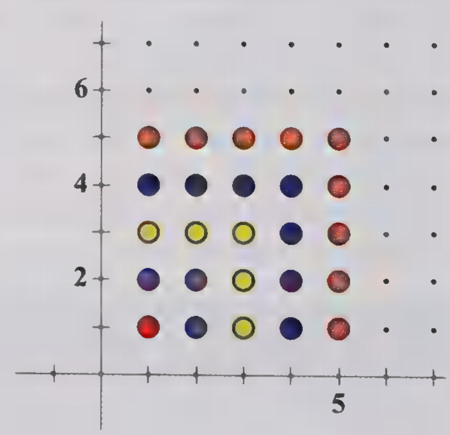
26. 15/74 day. Convert each given rate to the number of cisterns filled per day. We obtain 3, 1, 2/5, 1/3, and 1/5. We add these rates to find number of cisterns filled per day if all canals are open: $3 + 1 + 2/5 + 1/3 + 1/5 = 74/15$ cisterns per day, or 15/74 days per cistern. This problem is adapted from *Jiu zhang suan shu* (*Nine Chapters on the Mathematical Art*), a Chinese text in use by the beginning of the second century CE.

27. 330 different paths. John lives 11 blocks from Sam. Sam will travel south for 4 blocks and then east for 7 blocks. Visualize 11 blanks to be filled in with the letter S or the letter E. If four blanks are filled with S, then the remaining blanks are, by default, filled with E. So the correct number of paths is

$${}_{11}C_4 = \frac{11!}{4!7!} = \frac{11 \cdot 10 \cdot 9 \cdot 8}{4 \cdot 3 \cdot 2} = 330 \text{ paths.}$$

28. Problem of the Month: Readers are asked to encourage their students to submit solutions to this problem. Send solutions to Séan Madden (smadden@greeleyschools.org) or Ricardo Diaz (Ricardo.diaz@unco.edu) for possible publication in a future issue of *MT*.

29. 55/216. There are $6^3 = 216$ outcomes altogether. Let B represent the outcome on the blue die and let (R_1, R_2) represent the outcomes on the pair of red dice. The set of possible outcomes (R_1, R_2) can be visualized as a 6 × 6 square lattice in the coordinate plane. We develop a pattern to count the number of outcomes satisfying the specified condition. If $B = 1$, there are no such outcomes. If $B = 2$, there is one such outcome: $(R_1, R_2) = (1, 1)$. If $B = 3$, there are four such outcomes. Thus, the total number of favorable outcomes is $0^2 + 1^2 + 2^2 + 3^3 + 4^2 + 5^2 = 55$. The requested probability is 55/216. This problem was inspired by the game Risk.



30. 6. Only one possible necklace can have zero red beads, and only one can have zero black beads. Since a necklace can be turned over (reflected) and since beads can move over the clasp (rotated),

only one possible necklace can have 1 red bead and only one can have 1 black bead. Finally, two different necklaces can have 2 red beads; the red beads are either separated by black beads or strung consecutively.

31. 44. Between noon and 1:00 p.m., the hour and minute hands will be perpendicular twice. The first occurrence will be slightly past 12:15, and the second will be at about 12:48. (Keep in mind that as the minute hand advances, the hour hand does also, so it is not exactly on the 12.) Similarly, between the hours of 1:00 and 2:00, the hour and minute hands will be perpendicular twice. We might conjecture that the hands will be perpendicular twice in each span between consecutive hours, but this is not the case. The hands are perpendicular only once between 2:00 and 3:00, at about 2:27, since the next occurrence is at exactly 3 p.m. Thus, between 2:00 and 4:00, the hands form right angles only three times instead of the expected four times. Since the hands are perpendicular at exactly 9 p.m., there are only three such occurrences between 8:00 and 10:00. In every remaining hour span, there are two such occurrences, so we have a total of $12 \cdot 2 - 2 = 22$ times in a 12-hour period and 44 times in a 24-hour period.

Alternate solution 1: Start at 12:00. The angles traversed by the minute and hour hands, at time t in hours, are given by $\theta = 360°t$ and $\theta = 30°t$, respectively. The hands are perpendicular when the difference of the angles, $330°t$, is an odd multiple of 90°. The solutions are $330°t = 90°(1), 90°(3), 90°(5°), \ldots, 90°(87)$ to keep $0 \le t \le 24$. Thus, there are 44 occurrences.

Alternate solution 2: Let the minute hand and the hour hand be perpendicular to each other at a given time. The angular velocity of the minute and hour hands is 2π radians/hr. and $(\pi/6)$ radians/hr., respectively. The hands will once again be perpendicular when the minute hand has rotated π radians more than the hour hand; the first such time, t satisfies $2\pi t - (\pi/6)t = \pi$, giving $t = 6/11$ hr. In a 24-hr. period, the hands will be perpendicular $24/(6/11) = 44$ times.

Applications in Digital Im Processi

Jason Silverman, Gail L. Rosen, and Steve Essinger

Use digital signal processing to capitalize on an exciting intersection of mathematics and popular culture.

Students are immersed in a mathematically intensive, technological world. They engage daily with iPods, HDTVs, and smartphones—technological devices that rely on sophisticated but accessible mathematical ideas. In this article, we provide an overview of four lab-type activities that we have used successfully in high school mathematics classes to help motivate students to develop an interest in mathematics where their experiences intersect with the school curriculum.

Initial versions of these activities were piloted in an after-school mathematics enrichment program.

The revised versions, which are discussed here, were used by two second-year algebra teachers as part of their regular instructional time devoted to measures of central tendency, matrices and matrix operations, transformations, and piecewise defined functions. During our field testing, the activities were used as a real-world application after more-traditional instruction, but they can also be used to introduce the topics and to cultivate interest and increase motivation in prealgebra, algebra, and statistics courses.

Each activity begins with students working manually with small regions of an image (three or four pixel squares, with each pixel being represented by a cell in a matrix). Then, using specially designed applets, students quickly apply the same technique and reasoning to entire digital images. The initial exploration and calculations are used primarily to encourage students to slow down and understand what the cell values and matrices represent and how both connect to digital images—all of which they would likely miss if they jumped right to the applet. This understanding is critical for capitalizing on the mathematical potential of the remaining activities.

Each activity is discussed only briefly in this article, but detailed activity sheets and Web applets can be found at http://dk12.ece.drexel.edu/MT. In particular, we have not included the black-and-white activity in the article, but an introduction for both teachers and students at the site highlights the progression in detail.

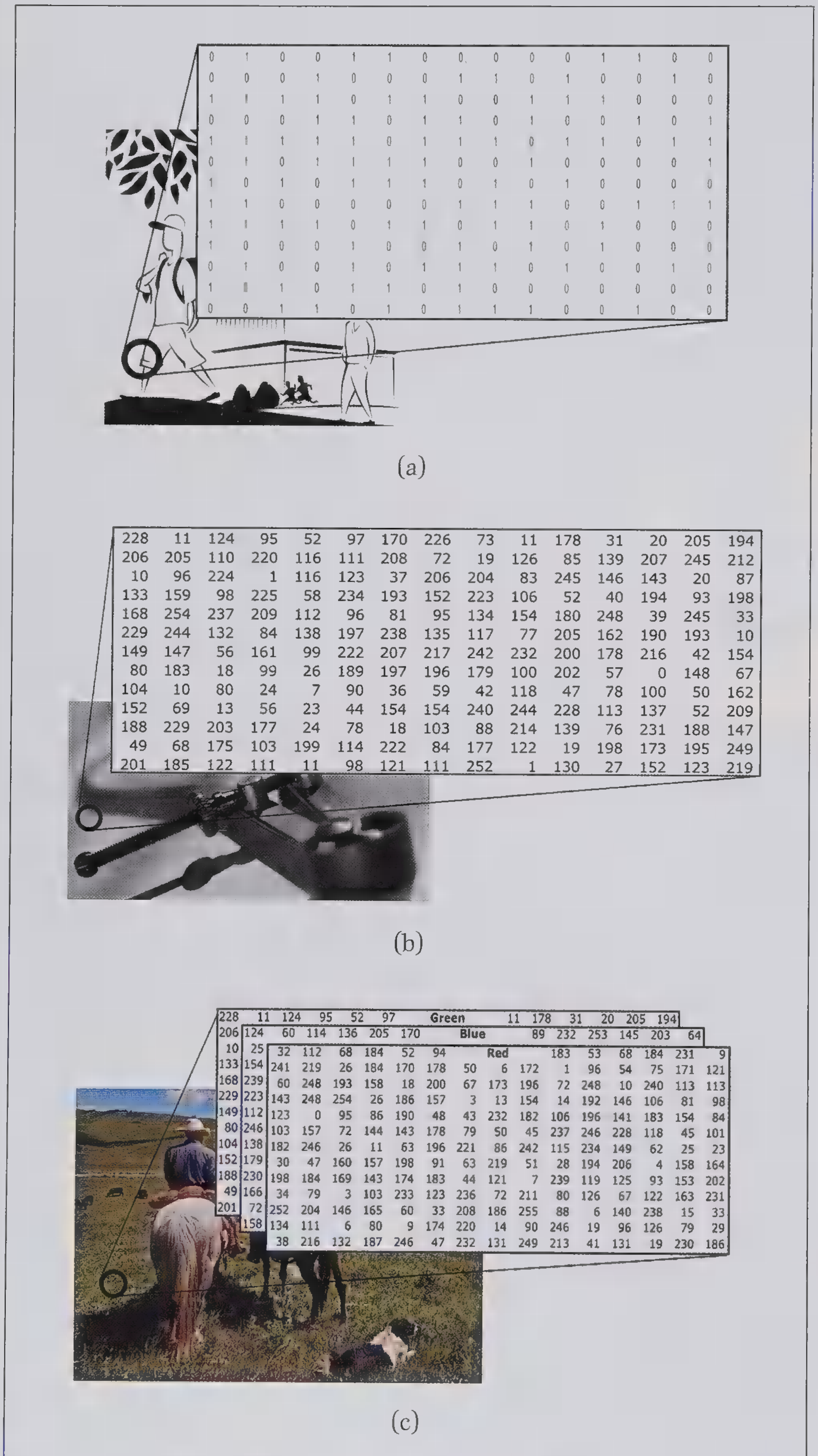


Fig. 1 Representations of a black-and-white image (**a**), a gray-scale image (**b**), and a color image (**c**) can be digital.

INTRODUCTION TO IMAGE PROCESSING

Digital signal processing (DSP)—the study of *signals* (observed phenomenon that can be mapped into discrete numerical representations) and analysis of these signals through mathematical methods—is a topic that can help students see the relevance of mathematics. Although the term *digital signal processing* may not be part of everyday language, DSP applications such as photo editing (e.g., red-eye reduction, image enhancement) and audio compression (e.g., .mp3 files on iPods, cell phone signals) are part of everyday life and popular culture. DSP applications can provide a context in which students can explore the mathematical ideas that lie behind digital image editing—such as removing backgrounds, adding tones to create particular effects (such as "sepia" to make an image appear aged), and superimposing images.

In digital image processing, images are divided into pixels (the number of pixels per inch depends on the resolution of the image). For a black-and-white image, each pixel is assigned a value of 0 for white (representing the complete absence of color) or 1 for black (representing full saturation of color), and each pixel value is recorded in a matrix (see **fig. 1a**). For a gray-scale image, each pixel is assigned a number between 0 and 255, where 0 represents the complete absence of black and 255 represents complete saturation of black (see **fig. 1b**). Color images can be thought of in a similar way: Each pixel is assigned a color that is defined using the red-green-blue (RGB) scale, meaning that each pixel has an assigned R, G, and B value between 0 (no color) and 255 (full saturation of color); the observed color is the result of the additive combination of the three.

For example, a pixel with RGB value [255, 0, 255] (red = 255, green = 0, blue = 255) would appear purple as a result of the combination of "pure" red and blue and the absence of green. A pixel with RGB value [215, 81, 23] (red = 215, green = 81, blue = 23) would appear deep orange as a result of a high red value mixed with much lower values of green and blue. In this way, a color image can be represented as three matrices, representing the intensity in each of red, green, and blue (see **fig. 1c**). Matrices of one of these forms are used throughout the image-processing lab activities.

The reduced cost of color imaging has rendered black-and-white and gray-scale images almost obsolete, but we have found that the progression delineated in these allows students to develop an understanding of the significance of the individual matrix elements and the relationship between the matrix and the image being represented. In this way, the meaning of connected R, G, and B matrices with values between 0 and 255 emerges from initial explorations of one matrix with values limited to 0 and 1.

LAB 1: INTRODUCTION TO DIGITAL IMAGES
Matrices and Piecewise Functions

To explore the RGB values of the individual pixels that comprise an image, students are introduced to the notion of a *pixel* and then explore the RGB values for individual pixels from various icons and images on their computer screens. Students use a digital color identification application, such as the DigitalColor Meter (see **fig. 2**). DigitalColor Meter is bundled as part of Apple OSX, and PC applications such as Pixie (http://www.nattyware.com/pixie.php) are available free on the Internet.

Students then are asked to think about the additive and subtractive models for creating colors by adding or removing specific amounts of red, green, and blue, a method commonly used in digital photography to add effects, correct overexposed and underexposed images, and increase image contrast.

Two methods of editing digital images include tinting and applying a thresholding operator. With tinting, the overall RGB values in an image are adjusted, resulting in an edited image that has an overall different color or feel. With thresholding, a threshold is chosen. Then, for each pixel in the image, if a specific value is larger than the threshold, that color is replaced with a 0 (i.e., the color is turned off); if the value is smaller than the threshold, that color is left unchanged. For example, **figure 3** shows a segment of the thresholding activities. Students find a "hidden" image by considering a threshold of 0.6; they shade all cells that have an intensity greater than 0.6 black and leave unshaded all cells that have intensity less that 0.6.

After tinting and thresholding manually, students use specifically developed applets for altering images using both methods. **Figure 4** shows how a subtractive model was used to tint a photograph of the planet Mars to make it appear redder. **Figure 5** shows how thresholding can increase contrast and overall image quality.

Throughout the activities, students are encouraged to move between the language of digital image editing and the language of mathematics, using such terms as *matrices*, *cells*, *matrix operations*, and *piecewise-defined functions*. For example, teachers can introduce the language and notation of piecewise-defined functions in this way: The threshold value is defined as

$$f(a_{ij}) = \begin{cases} a_{ij}, \text{ if } a_{ij} \le T \\ 0, \text{ if } a_{ij} > T \end{cases},$$

where a_{ij} denotes the R, G, or B value for the cell in the ith row and jth column and T denotes the threshold value.

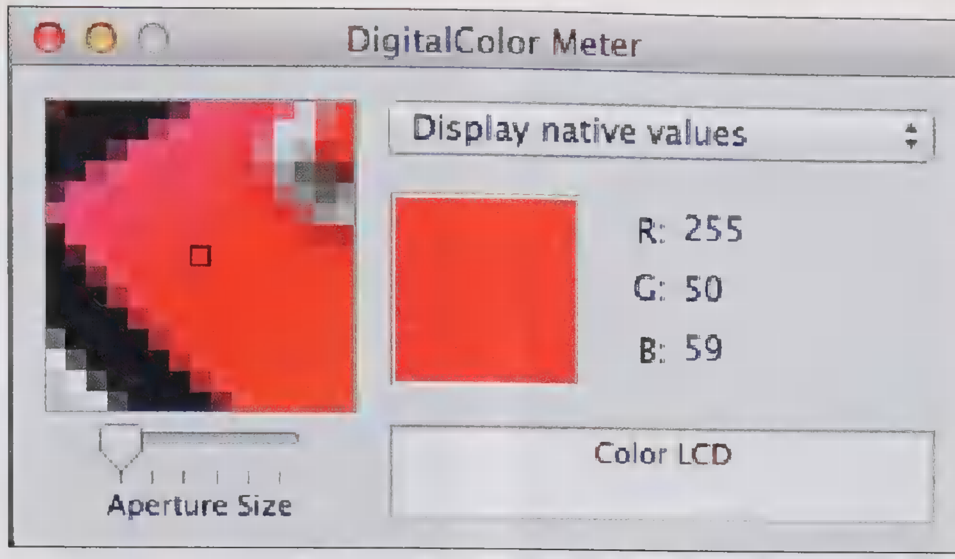


Fig. 2 This digital color meter is bundled with Apple OSX.

0.532	0.312	0.364	0.458	0.479	0.371	0.412	0.562	0.411
0.453	0.794	0.651	0.401	0.446	0.497	0.786	0.636	0.443
0.403	0.774	0.675	0.415	0.312	0.462	0.789	0.629	0.479
0.488	0.534	0.512	0.534	0.493	0.570	0.409	0.534	0.493
0.452	0.470	0.531	0.821	0.763	0.662	0.578	0.554	0.583
0.603	0.297	0.361	0.258	0.762	0.417	0.425	0.515	0.707
0.572	0.654	0.584	0.450	0.456	0.443	0.511	0.654	0.427
0.301	0.484	0.899	0.632	0.783	0.611	0.791	0.507	0.413
0.488	0.534	0.512	0.534	0.493	0.570	0.409	0.534	0.493

Fig. 3 Thresholding can be used to find a hidden image.

LAB 2: IMAGE "DENOISING"
Measures of Central Tendency

Elementary school and middle school students learn about measures of central tendency and how to calculate them but seldom consider the question of what this calculation represents. In this lab, students explore an interesting application of measures of central tendency—namely, how application of two measures of central tendency (the mean and the median) can be used to improve the quality of "corrupted" digital images.

Students are first introduced to various types of "noise" that can occur in digital photography when one or many pixels become "corrupt" (i.e., the RGB values do not accurately depict the event captured in the image) as a result of faulty hardware or environmental conditions. Students explore two specific types of noise: (1) "dead" pixels, which cause either specific or random pixels to be a constant color (often called salt-and-pepper noise); and (2) grainy images, such as those taken in low-light environments. In

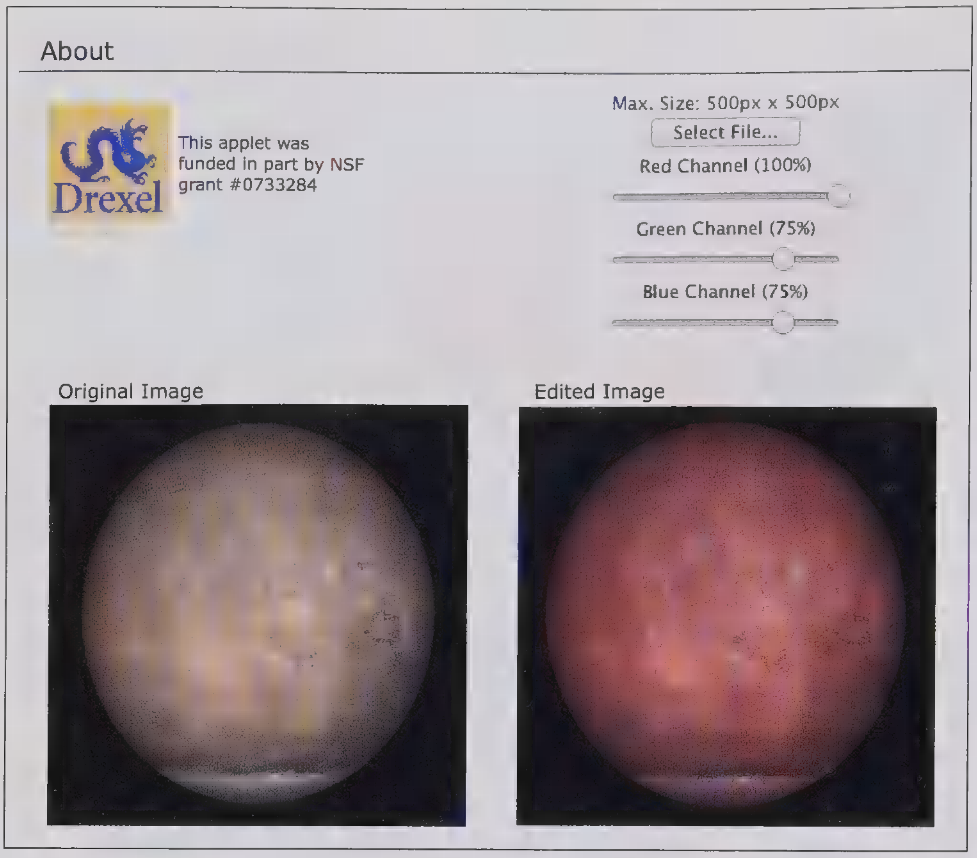


Fig. 4 A subtractive color model interface makes Mars appear as the "red" planet.

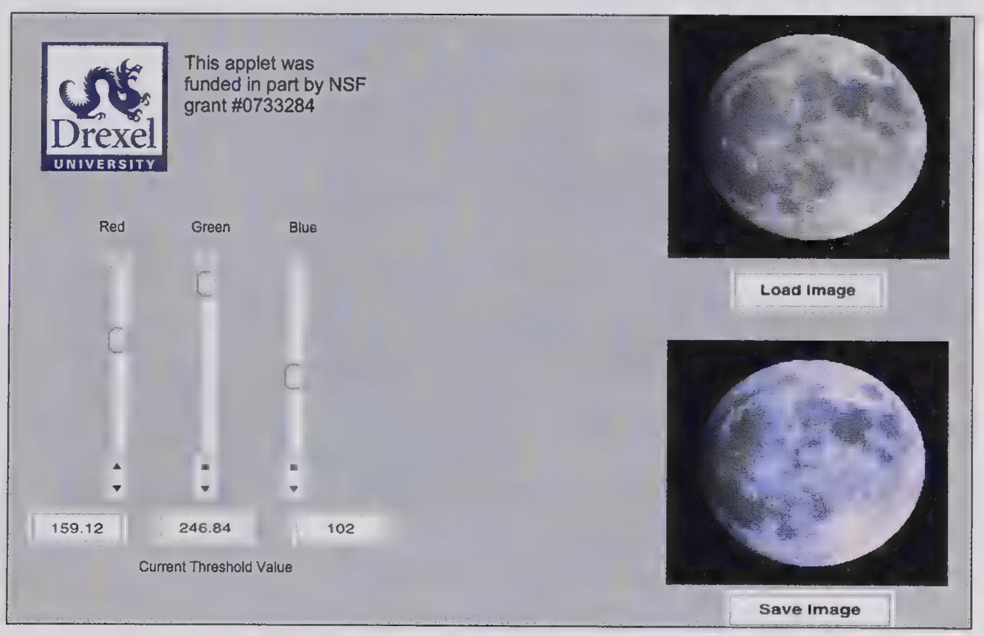


Fig. 5 Adjusting the color channels of an image can give the perception of contrast.

255	255	255
255		0
255	0	0

(a)

240	255	229
229		50
248	231	221

(b)

Fig. 6 Students practice "smoothing" by filling in the central blank with the mean (**a**) or the median (**b**) of the surrounding eight pixels.

exploring these two types of noise, students develop an understanding of how the mean and the median can help identify and correct corrupt pixels.

To remove noise, students learn about *smoothing*, in which the value of each pixel is replaced with the mean or the median of the eight pixels surrounding it. Students begin by practicing manual smoothing, using both the mean and the median, for a variety of 3×3 matrices (portions of an image) (see **fig. 6**). For example, as shown in **figure 7a**, smoothing by using the mean would result in a blurred boundary (the center pixel would have a value of 159), whereas smoothing by using the median would result in a crisp boundary (the center pixel would have a value of 255). Either version would be an improvement over a dead pixel. There is a trade-off, however: When either smoothing technique is used, some of the fine detail of the image will be lost because even pixels that are not corrupt are smoothed.

Students are then introduced to the denoising applet that allows them to complete mean or median smoothing for each pixel in an image. Using this applet, students can upload images corrupted with either type of noise mentioned above and can test both the mean and the median smoothing algorithms (see **fig. 7b**). Students observe that the median filter provides superior correction on the salt-and-pepper image and can discuss why, referring to the differences between the two measures.

Finally, students discuss the trade-off of denoising versus blurring the image and make conjectures about when photo editing software might be used to improve image quality.

These activities and discussions provide students with a context to explore the significance and meaning of mean and median and can ultimately help connect procedural and conceptual understanding of these ideas.

LAB 3: IMAGE CONTRAST

Linear, Power, and Log Transformations

In the secondary school algebra curriculum, students solve and graph equations, but often their understanding of equations is limited. Thompson (1994) notes that many students think of equations in a rudimentary way—as two expressions separated by an equals sign and without any notion of quantities or relationships. This activity underscores the notion of function as a relationship between a domain and a range and enriches students' ability to interpret equations and graphs.

Students explore linear, square, square root, exponential, or natural logarithm transformations of the intensity value for each pixel in the image. They begin by calculating by hand various transformations for 3×3 portions of an image.

Figure 8a depicts an original square of pixels with their blue intensity value. Students apply a quadratic transformation (see **fig. 8b**) and a linear transformation (see **fig. 8c**) of the intensity value for each pixel (all transformed values larger than 255 are represented with full saturation [255]). Students then discuss the benefits of each type of transformation, including the fact that the linear transformation affects each number in a constant way, whereas the quadratic transformation affects smaller and larger values differently.

After exploring the ideas manually, students use another applet that allows them to apply specific transformations to each pixel of an uploaded image of their choice. This applet then allows for the image to be "normalized," an additional step that takes into account instances in which many of the resulting intensity values are higher than 255 (remember, 255 is complete saturation of color, so the same color is displayed whenever the value is greater than or equal to 255). To normalize the image, students create a new unit that is 1/255 as large as the highest value and then calculate the value of each pixel in terms of that unit. **Figure 9b** shows that the highest value of this particular image is 12544 and that 1/255 of 12544 is approximately 49.2. Each value is then divided by 49.2 to normalize it and represent the color level in terms of the new unit (see **fig. 9c**).

Follow-up discussions can build on students' experiences with Photoshop or other digital photo editing applications, including similarities and differences between resulting images, how to improve contrast to increase the clarity of an image, how the various nonlinear operators improved the clarity of the image, and conjectures about the characteristics of images that would be improved for each operator. These transformations, which provide a proactive response to "When will I ever use quadratics or logs?" allow students to begin to see the mathematical ideas that lie behind image enhancement in fields from astronomy to medical imaging and military science—applications that directly affect their lives.

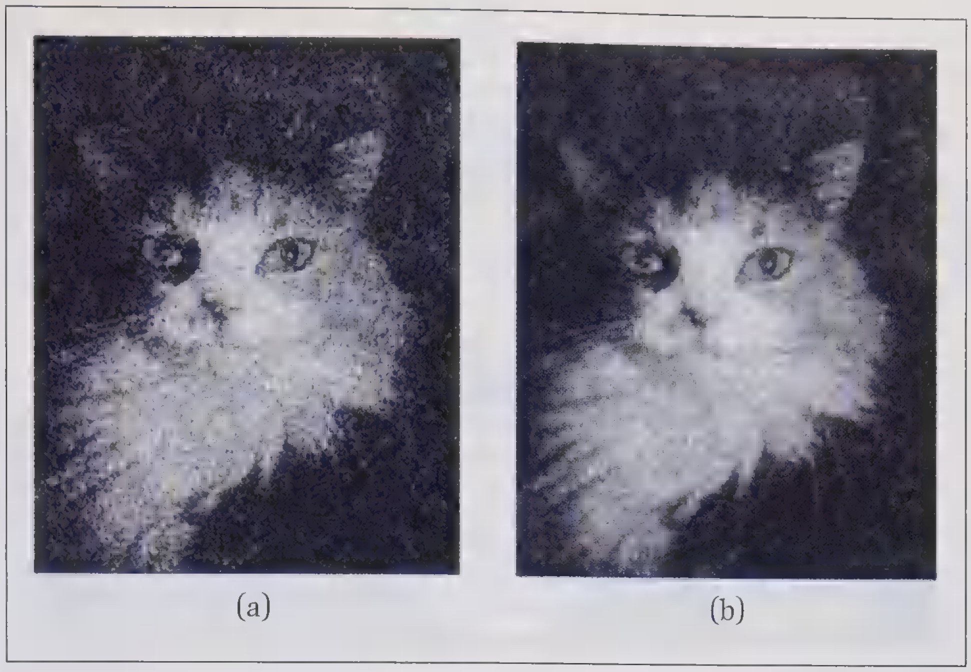


Fig. 7 Smoothing by using the mean (**a**) or the median (**b**) can be used to improve the image.

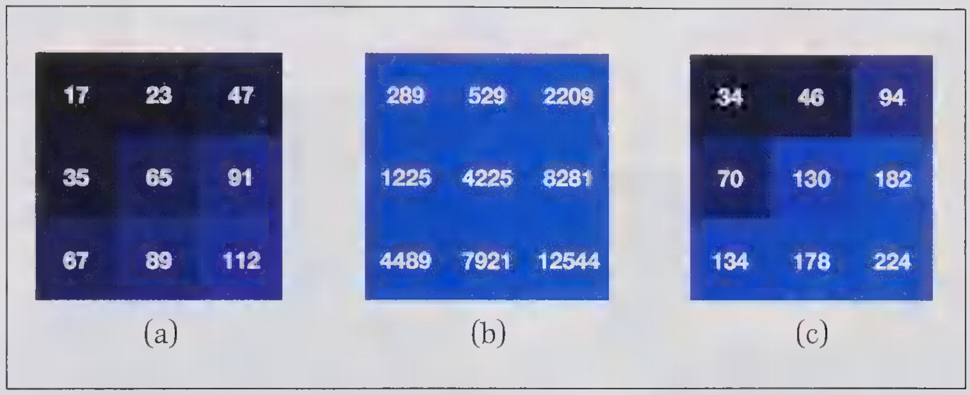


Fig. 8 An original 3×3 portion of an image (**a**) is adjusted first by a quadratic transformation (**b**) and then by a linear transformation (**c**).

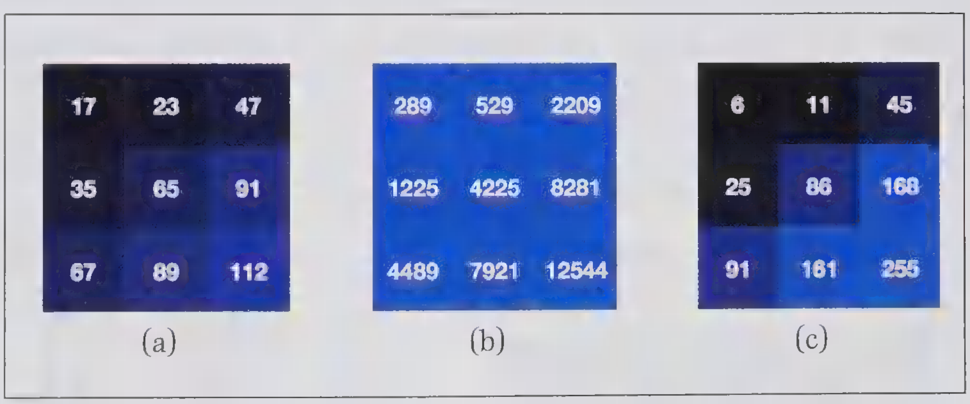


Fig. 9 Applying a quadratic transformation to the image in (**a**) yields the image in (**b**); the image in (**c**) shows the normalized quadratic transformation.

LAB 4: EDGE DETECTION

Rate of Change and Gradient

Edge detection in digital image processing involves a number of methods used to identify areas in an image in which the brightness or intensity changes significantly. These locations are likely to correspond to discontinuities in depth, discontinuities in orientation, changes in material, or variations in illumination—edges between different objects.

Several highly sophisticated algorithms are used for edge detection, but, at its most basic, edge detection is about calculating the change in intensity from pixel to pixel and comparing this change with a threshold value. If the change is big enough, then that location is defined to be an edge. If the change is not big enough, there is no edge at that location. As with the previous activities, students are given a paper-and-pencil task before using the applets (see **fig. 10**).

This task asks them to examine a color-intensity matrix, calculate the difference between pairs of adjacent cells, and compare this difference with a specified threshold. If the difference is greater than the threshold, the rightmost cell in the pair is

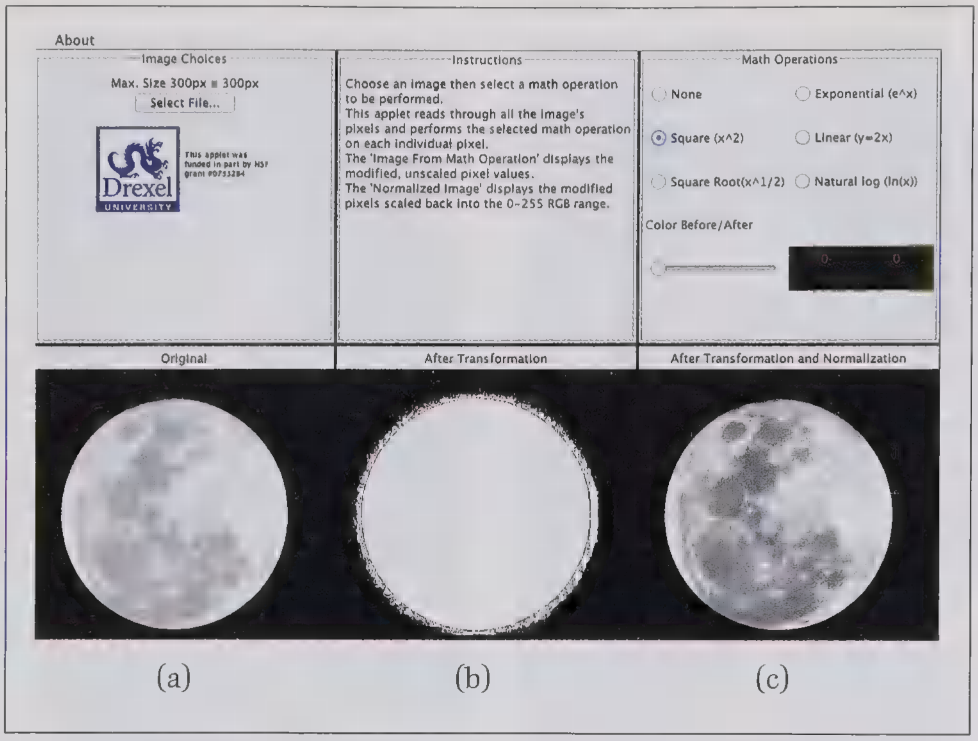


Fig. 10 The transformation applet shows how the image changes.

(a)

45	55	105	98	95
102	21	18	40	45
95	100	110	155	162
15	105	140	110	125

(b)

45	55	105	98	95
102	21	18	40	45
95	100	110	155	162
15	105	140	110	125

Fig. 11 The edge matrix generated by students (**a**) can be compared with the actual intensity matrix (**b**).

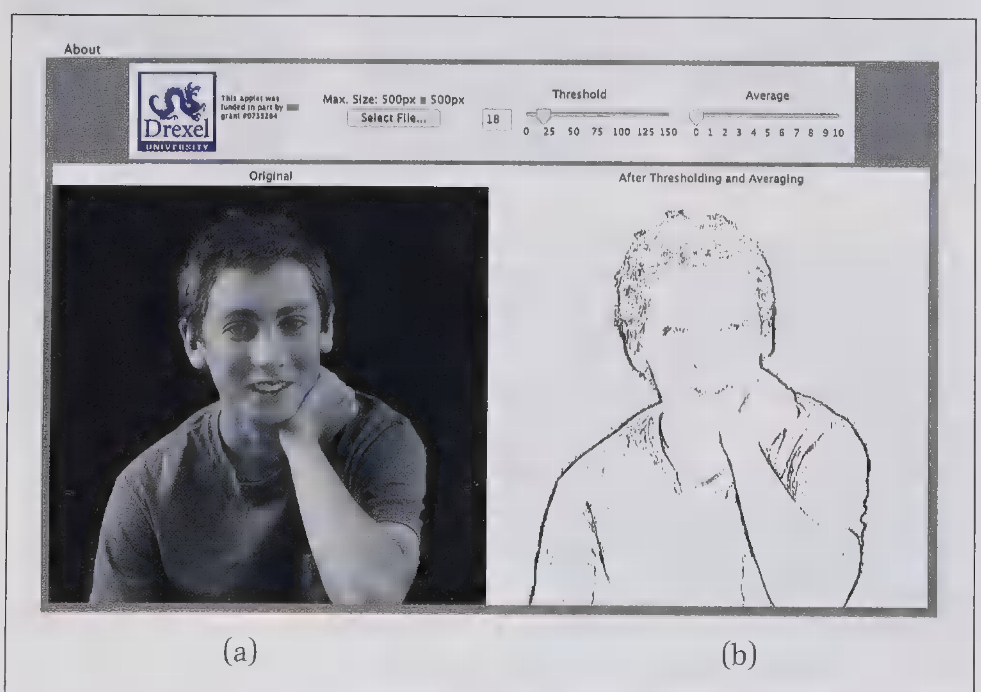


Fig. 12 Students can explore a variety of image changes by using the edge-detection applet.

colored; if the difference is less than the threshold, the cell is left uncolored.

Figure 11a shows a type of edge matrix created by students in which the colored cells indicate sharp changes in color intensity. Students can then compare the matrix they created with the actual intensities (**see fig. 11b**) and confirm the sharp changes in color intensity. In addition, they are likely to observe that their procedure does not identify sharp changes in intensity above or below a given cell, and they can discuss how the procedure to identify edges might be adapted or refined.

After working through the edge detection by hand, students then use an applet to perform a similar task on an image of their choice. This applet allows them to define a threshold and explore how the edges identified by the edge-detection algorithm change, depending on the threshold and the original image (see **fig. 12**). After completing the activity, students discuss appropriate threshold values for particular situations and applications of this feature in digital photo editing applications, including green-screen production, when portions of one video are superimposed over another video (as when the image of a meteorologist is superimposed over a weather map).

RESULTS AND CONCLUSION

We have found that DSP, particularly image processing, can involve students in doing real mathematics. After implementing these four activities in two first-year algebra classes as part of the regular instructional program, we analyzed pre- and post-unit surveys designed to measure students' interest in mathematics, feelings of self-efficacy for mathematics, and mathematical beliefs. The data showed significant gains in students' personal connection with mathematics, their beliefs about potential careers that require mathematics, and their enjoyment of mathematics.

Clearly, DSP applications and activities provide a meaningful context for students to engage with legitimate mathematics and connect their personal lives with school curricula. Further, students reported that they enjoyed the process.

We encourage readers to explore these activities and resources and consider using them to highlight the connections between cutting-edge engineering applications, mathematics content, and today's students.

ACKNOWLEDGMENTS

Preparation of this article was supported by National Science Foundation Grant no. DRL 0733284. Any opinions, findings, conclusions, or recommendations expressed here are those of the authors and do not necessarily reflect the views of the National Science Foundation.

BIBLIOGRAPHY

Ames, Carole. 1992. "Classrooms: Goals, Structures, and Student Motivation." *Journal of Educational Psychology* 84 (3): 261–71.

Barrow, Harry G., and Jay M. Tenenbaum. 1981. "Interpreting Line Drawings as Three-Dimensional Surfaces." *Artificial Intelligence* 17 (1–3): 75–116.

Bremigan, Elizabeth George. 2003. "Developing a Meaningful Understanding of the Mean." *Mathematics Teaching in the Middle School* 9 (1): 22–26.

Chazan, Daniel C. 2000. *Beyond Formulas in Mathematics and Teaching: Dynamics of the High School Algebra Classroom.* Series on School Reform. New York: Teachers College Press.

Duda, Joan L., and John G. Nicholls. 1992. "Dimensions of Achievement Motivation in Schoolwork and Sport." *Journal of Educational Psychology* 84 (3): 290–99.

Mathematics Learning Study Committee and National Research Council. 2001. *Adding It Up: Helping Children Learn Mathematics.* Washington, DC: National Academies Press.

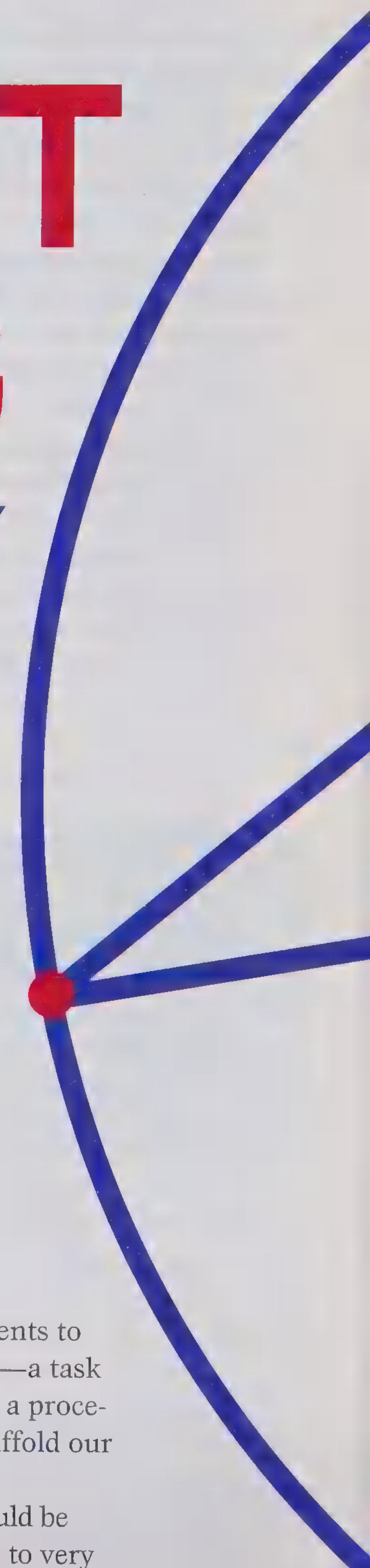
Silverman, Jason, and Gail L. Rosen. 2010. "Supporting Students' Interest in Mathematics through Applications from Digital Image Processing." *Journal of the Research Center for Educational Technology* 6 (2): 63–77.

Thompson, Patrick W. 1994. "Students, Functions, and the Undergraduate Curriculum." In *Research in Collegiate Mathematics Education I: Issues in Mathematics Education*, edited by Ed Dubinsky, Alan Schoenfeld, and Jim Kaput, vol. 4, pp. 21–44. CBMS Issues in Mathematics Education Series. Providence, RI: American Mathematical Society and Conference Board of the Mathematical Sciences.



JASON SILVERMAN, silverman@drexel.edu, is an associate professor of mathematics education at Drexel University in Philadelphia. His interests include supporting teachers' development of Mathematics Content Knowledge for Teaching and working with the Math Forum @ Drexel. **GAIL L. ROSEN,** gailr@ece.drexel.edu, is an assistant professor of electrical and computer engineering at Drexel University. Her research interests include signal processing, bioinformatics, and bringing her research to the K–12 classroom. **STEVE ESSINGER,** sessinger@gmail.com, is a doctoral candidate in electrical and computer engineering at Drexel University. His research interests include signal processing, machine learning, and computational biology.

IMPROVING STUDENT REASONING IN GEOMETRY

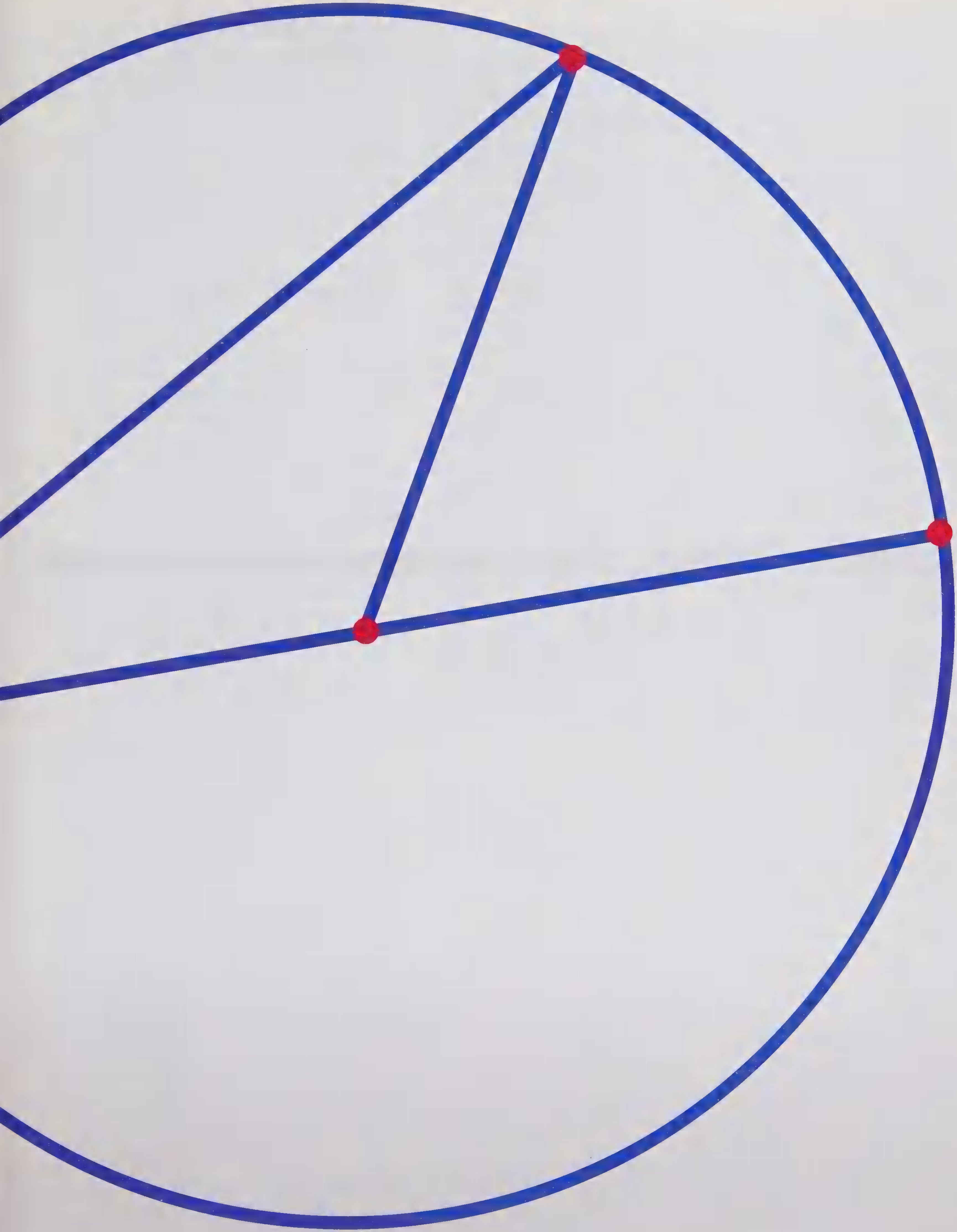


Parallel geometry tasks with four levels of complexity involve students in writing and understanding proof.

Bobson Wong and Larisa Bukalov

In our years of teaching geometry, the greatest challenge has been getting students to improve their reasoning. Many students have difficulty writing formal proofs—a task that requires a good deal of reasoning. Proof is a problem-solving activity, not a procedure that can be done routinely (Cirillo 2009). We wanted to find ways to scaffold our instruction to prepare students for harder problems.

In planning geometry lessons, we noticed that many problems that we selected could be arranged from very straightforward (often a simple diagram with a missing quantity) to very complex (such as a detailed formal proof). Tiering the lessons—that is, creating multiple pathways for students to understand the goals of a lesson—might be the best strategy (Pierce and Adams 2005). The work of van Hiele (1986) and others building on van Hiele's work, such as Burger and Shaughnessy (1986) and Clements (2003), was extremely helpful in suggesting that geometry requires higher-order thinking and that students need more experience with lower levels of thinking before they can succeed at higher levels. Following the recommendation of Artzt et al. (2008), we wanted to present problems in a way that was accessible enough for students to use their prior knowledge but also challenging enough so that they could extend their learning.



Although sequencing problems properly is important, we knew that sequencing alone would not guarantee that all students would be able to improve their reasoning. We wanted all students to have an equal opportunity to improve, and we wanted to avoid the dangers of "tracking" students by ability. In tracked classes, lower-level students often have limited exposure to a high-quality mathematics education (Useem 1990). We wanted a system that was fair and flexible.

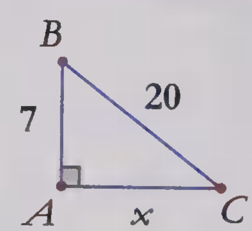
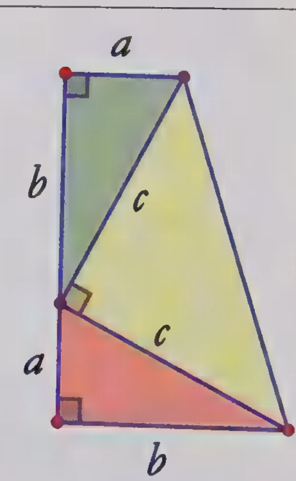
Our solution was to divide the lessons into parallel tasks, allowing students with different levels of understanding of a topic to work on the same task simultaneously (Small and Lin 2010). We organized problems into four levels of complexity but allowed students to select their own level and move freely between levels. By having access to all levels, students could monitor their own progress and know what they needed to do to move to the next level. We fit our lessons within a familiar three-part framework: whole-group introductory discussion, guided independent practice, and whole-group summary. At the same time, our model was simple enough for teachers, students, and parents to understand.

THE FOUR LEVELS OF COMPLEXITY

Our organizational model divides problems into four levels with questions of increasing complexity. These four levels were inspired by the four-point system used to report our statewide exam questions as well as the four Depth of Knowledge levels (Webb, Vesperman, and Ely 2005). **Table 1** contains a brief, general description of levels of problems in our model, along with a sample problem for each level for a lesson on the Pythagorean theorem.

Level 1, the simplest level, consists of problems that can be solved by directly applying a fact, method, or formula. Problems at this level typically do not require students to use precise mathematical language. For example, the level 1 problem in **table 1** is a straightforward application of the Pythagorean theorem. From the diagram, students can immediately identify the two legs and the hypotenuse of the triangle and apply the formula $a^2 + b^2 = c^2$ without having to name the theorem or explain why triangle *ABC* is a right triangle. Other examples of level 1 problems include finding the midpoint of a line segment given the coordinates of its endpoints or stating the properties of a parallelogram.

Table 1 Levels of Complexity for Problems

Level	Characteristics	Sample Problem for Lesson on Pythagorean Theorem
1	Student solves problem by directly recalling a fact, method, or formula.	Find the value of x in simplest radical form. 
2	Student solves problem through an additional step beyond a level 1 problem. Typically, some guidance about the method needed for solving is provided.	The lengths of three sides of a triangle are 25, 7, and 24. Determine whether the triangle is a right triangle.
3	Student solves problem by selecting the appropriate pieces of information independently (usually two or more definitions, theorems, formulas, or methods).	In an isosceles trapezoid, the lengths of the bases are 14 in. and 30 in. The length of each of the nonparallel sides is 10 in. Find the length of the altitude of the trapezoid.
4	Student solves problem by using deductive reasoning to prove mathematical statements.	Explain how the diagram at right can be used to prove the Pythagorean theorem. 

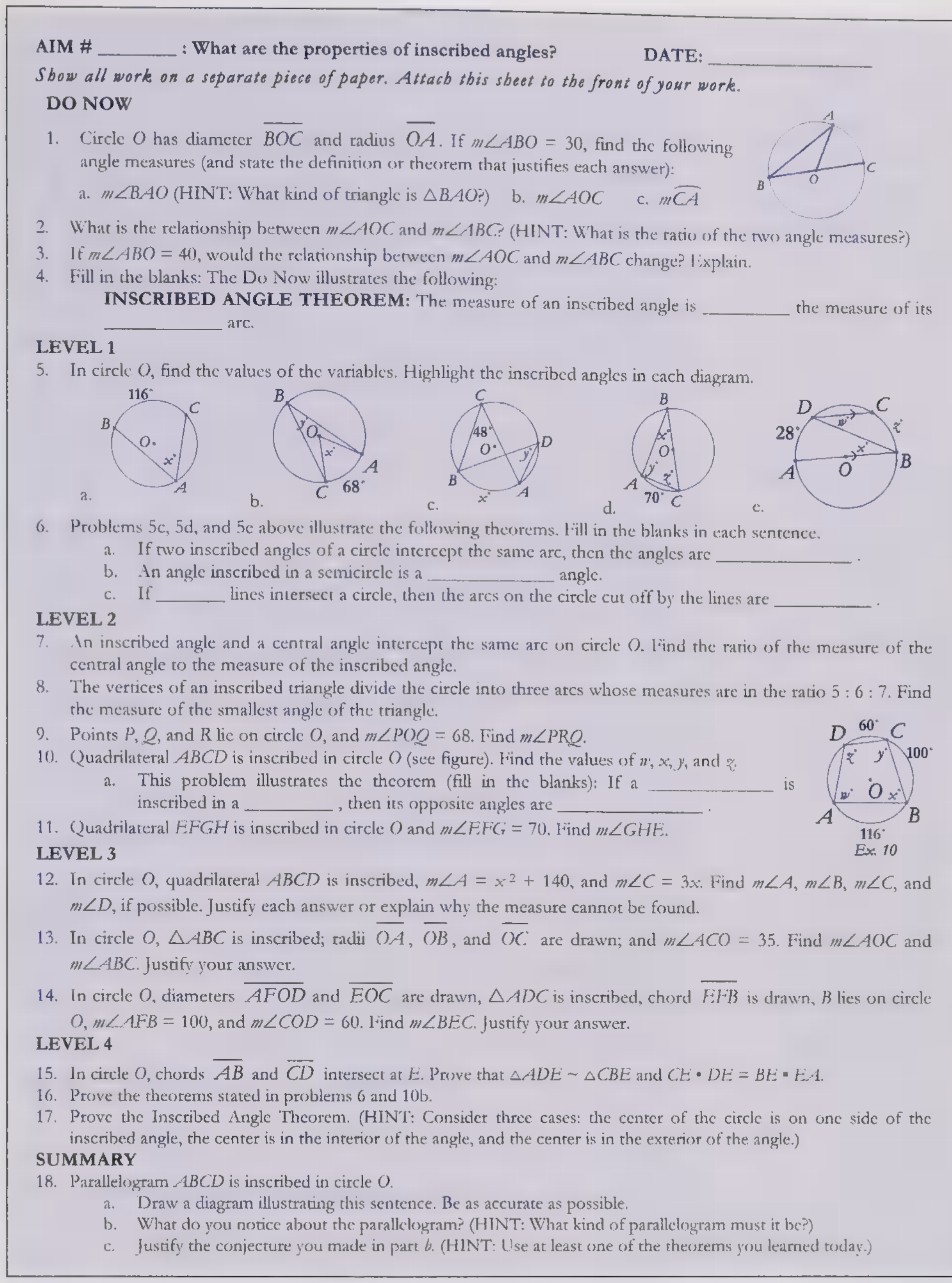
Level 2 problems require an additional step beyond a level 1 problem. For example, level 2 problems may require students to draw an accurately labeled diagram when none is provided. A level 2 problem may also require translating a simple word problem into an algebraic representation. To solve the level 2 problem shown in **table 1**, students must recognize that if the given triangle is right, then the side length of 25 must be the hypotenuse, the longest side of a right triangle. Level 2 problems typically provide some instruction about the method required to solve the problem, such as "Determine whether the triangle is a right triangle" or "Use the midpoint formula to determine whether the quadrilateral is a parallelogram."

Level 3 problems require students to determine independently what information is needed for a solution—typically, several formulas, theorems, or facts. The level 3 question in **table 1** requires students to integrate several ideas: drawing an appropriately labeled diagram, recognizing that the altitudes divide the trapezoid into right triangles and a rectangle, and then applying the Pythagorean theorem and properties of rectangles to find the length of the altitude. Level 3 problems can also include interpreting complex multistep diagrams, a task that requires the application of several theorems. Problems at this level do not require applying deductive reasoning to write formal Euclidean proofs. However, level 3 problems can ask students to select appropriate solution methods and justify their calculations by citing appropriate definitions or theorems. For example, a level 3 problem could give the coordinates of a quadrilateral and ask students to prove that it is a parallelogram.

Level 4 problems require students to use deductive reasoning to prove mathematical statements. Students must have reasoning skills that are strong enough for writing formal proofs. Problems at this level include formal Euclidean proofs.

SAMPLE LESSON WITH PARALLEL TASKS: INSCRIBED ANGLES

By dividing classwork into four levels, we provide multiple entry points for students. Problems from all four levels were included on one activity sheet distributed to all students so that they could see questions from all levels. However, students would have to master one level before moving on to the next. Following Small and Lin's (2010) advice, we allow students to select the level that they feel is most appropriate for their readiness for the lesson. We believe that allowing students to select the appropriate starting point for their work empowers them. Many weaker students told us that they did not feel stigmatized by starting at lower levels because they were able to get more practice to work up to higher levels. Although many students who selected a level beyond their understanding for that topic soon chose a lower level, others found that they could handle a more challenging level than they originally thought. Whenever possible, we encourage students to start at a higher level if they find the lower-level questions too easy.



AIM # ______ : What are the properties of inscribed angles? **DATE: ____________**

Show all work on a separate piece of paper. Attach this sheet to the front of your work.

DO NOW

1. Circle O has diameter $\overline{BOC}$ and radius $\overline{OA}$. If $m\angle ABO = 30$, find the following angle measures (and state the definition or theorem that justifies each answer):
   a. $m\angle BAO$ (HINT: What kind of triangle is $\triangle BAO$?) b. $m\angle AOC$ c. $m\widehat{CA}$
2. What is the relationship between $m\angle AOC$ and $m\angle ABC$? (HINT: What is the ratio of the two angle measures?)
3. If $m\angle ABO = 40$, would the relationship between $m\angle AOC$ and $m\angle ABC$ change? Explain.
4. Fill in the blanks: The Do Now illustrates the following:
   INSCRIBED ANGLE THEOREM: The measure of an inscribed angle is ________ the measure of its __________ arc.

LEVEL 1

5. In circle O, find the values of the variables. Highlight the inscribed angles in each diagram.
   a. b. c. d. e.
6. Problems 5c, 5d, and 5e above illustrate the following theorems. Fill in the blanks in each sentence.
   a. If two inscribed angles of a circle intercept the same arc, then the angles are ____________ .
   b. An angle inscribed in a semicircle is a ____________ angle.
   c. If _______ lines intersect a circle, then the arcs on the circle cut off by the lines are _________ .

LEVEL 2

7. An inscribed angle and a central angle intercept the same arc on circle O. Find the ratio of the measure of the central angle to the measure of the inscribed angle.
8. The vertices of an inscribed triangle divide the circle into three arcs whose measures are in the ratio 5 : 6 : 7. Find the measure of the smallest angle of the triangle.
9. Points P, Q, and R lie on circle O, and $m\angle PQQ = 68$. Find $m\angle PRQ$.
10. Quadrilateral $ABCD$ is inscribed in circle O (see figure). Find the values of w, x, y, and z.
    a. This problem illustrates the theorem (fill in the blanks): If a ____________ is inscribed in a _________ , then its opposite angles are ______________ .
11. Quadrilateral $EFGH$ is inscribed in circle O and $m\angle EFG = 70$. Find $m\angle GHE$.

Ex. 10

LEVEL 3

12. In circle O, quadrilateral $ABCD$ is inscribed, $m\angle A = x^2 + 140$, and $m\angle C = 3x$. Find $m\angle A$, $m\angle B$, $m\angle C$, and $m\angle D$, if possible. Justify each answer or explain why the measure cannot be found.
13. In circle O, $\triangle ABC$ is inscribed; radii $\overline{OA}$, $\overline{OB}$, and $\overline{OC}$ are drawn; and $m\angle ACO = 35$. Find $m\angle AOC$ and $m\angle ABC$. Justify your answer.
14. In circle O, diameters $\overline{AFOD}$ and $\overline{EOC}$ are drawn, $\triangle ADC$ is inscribed, chord $\overline{EFB}$ is drawn, B lies on circle O, $m\angle AFB = 100$, and $m\angle COD = 60$. Find $m\angle BEC$. Justify your answer.

LEVEL 4

15. In circle O, chords $\overline{AB}$ and $\overline{CD}$ intersect at E. Prove that $\triangle ADE \sim \triangle CBE$ and $CE \cdot DE = BE \cdot EA$.
16. Prove the theorems stated in problems 6 and 10b.
17. Prove the Inscribed Angle Theorem. (HINT: Consider three cases: the center of the circle is on one side of the inscribed angle, the center is in the interior of the angle, and the center is in the exterior of the angle.)

SUMMARY

18. Parallelogram $ABCD$ is inscribed in circle O.
    a. Draw a diagram illustrating this sentence. Be as accurate as possible.
    b. What do you notice about the parallelogram? (HINT: What kind of parallelogram must it be?)
    c. Justify the conjecture you made in part *b*. (HINT: Use at least one of the theorems you learned today.)

Fig. 1 This sample activity sheet includes questions from all four levels.

To give a better idea of what typical level 4 classwork looks like, we have included a sample activity sheet (see **fig. 1**). Samples of student work are also included to illustrate the level of detail and type of thinking expected at each level. This activity sheet provides not only practice problems but also enough guided questions (such as nos. 4, 6, and 10) so that students can learn new information without much direct instruction. Students who started at levels 3 or 4 were encouraged to review the lower-level problems to make sure that they knew how to do them and get relevant information (such as theorems and formulas) for higher-level problems.

To help students work independently and make the class flow more smoothly, we designed the

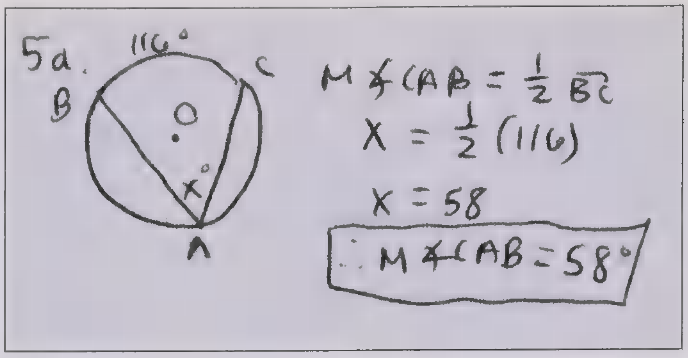


Fig. 2 A student response to problem 5(*a*) involves only applying the appropriate theorem and performing the calculation without justification.

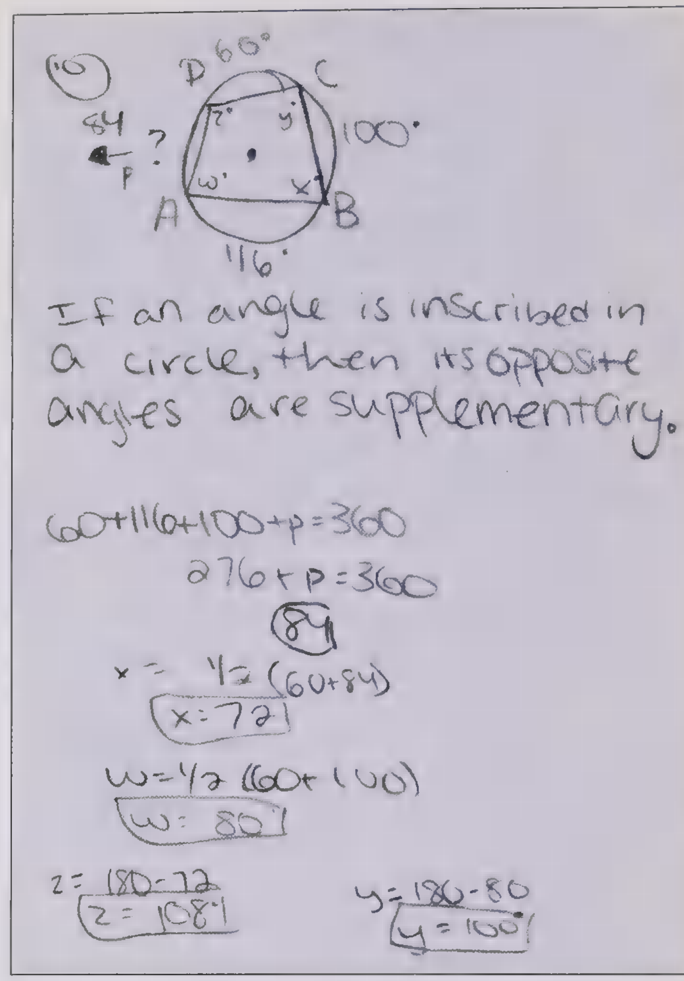


Fig. 3 This level 2 question requires students to draw the diagram and make conjectures. Note the student's vocabulary error.

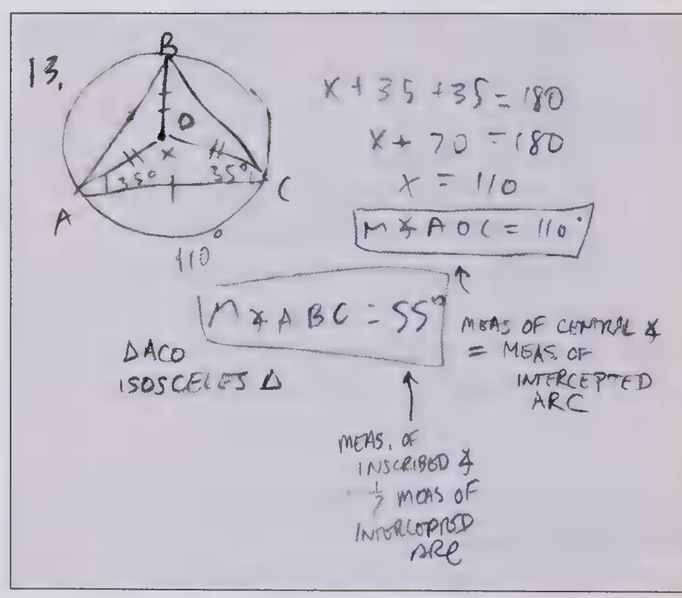


Fig. 4 Answering question 13 requires applying more than one theorem, a characteristic requirement of a level 3 question.

tasks so that each student worked on the same topic simultaneously. This approach allowed all students to participate in a common discussion at the beginning and the end of class. We also posted hints and answers to problems both on the board and online so that students could check their work themselves in class or at home. In addition, we found that students working at a lower level sought help from others who had already completed those problems or were working on a higher level, thus freeing us to circulate around the room and help individual students more.

The Do Now portion of the activity sheet, which all students complete at the beginning of class, reviews relevant prior knowledge and introduces the new material in the lesson. In the example shown in **figure 1**, the Do Now task reviews previously learned theorems about the relationship between central angles and arcs and elicits the inscribed angle theorem from numerical examples and a fill-in-the-blank statement. This section is easy enough for students at all levels to complete and also provides enough additional information for them to start the new work.

The next problems are divided into four levels of complexity. In general, the lesson's most basic concepts are introduced in the Do Now assignment and level 1 questions; more advanced concepts are presented in levels 2 and 3; and the concepts required for full mastery are given in level 4. Each level contains practice questions appropriate for that level of difficulty. Each level can also contain enrichment questions or questions that introduce more difficult concepts for the next level.

Level 1 questions contain direct applications of the inscribed angle theorem, which was introduced in the Do Now task. **Figure 2** shows student work for problem 5(*a*)—a straightforward calculation applying the theorem with no explanation required. Following the advice of Cirillo (2009), we wanted to encourage students at all levels of understanding to make conjectures. Thus, this level contains a summary question (no. 6) that requires students to explain in words what they see in the examples by filling in blanks in sentences. This question also introduces theorems used for other levels without requiring students to prove them.

Level 2 questions require the additional step of translating words into algebraic expressions or appropriately labeled diagrams. For example, question 9 is similar to 5(*b*), a level 1 problem, but lacks a diagram. Question 10 allows students to make further conjectures that lead to a theorem (see **fig. 3**).

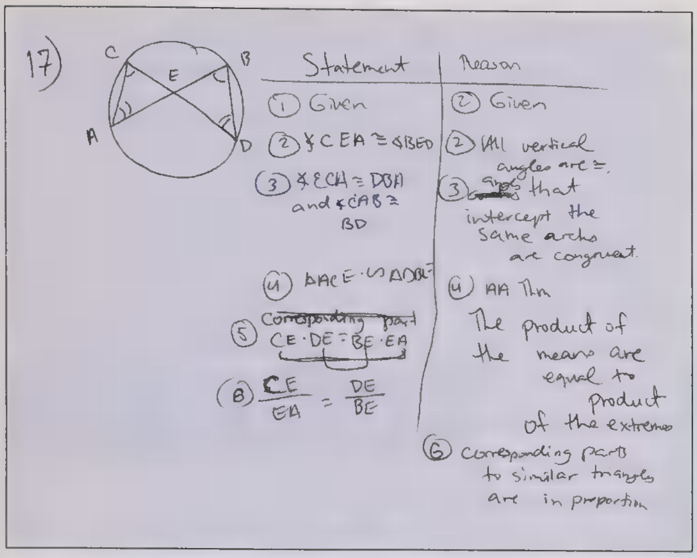


Fig. 5 Although this student has the right general idea, a letter is missing in statement 3, and the proportion is wrong in statement 6. Most teachers would insist that statement 6 precede statement 5.

Level 3 questions require students to use several theorems, formulas, or ideas in the same problem. For example, to answer question 13, students must apply the inscribed angle theorem as well as other theorems. Students can justify their answers to level 3 questions by annotating their work with appropriate definitions or theorems, as shown in the student work in **figure 4**.

Level 4 questions help students summarize the lesson by asking them to prove the theorems elicited in the previous levels. **Figure 5** shows an example of a student's formal proof for question 17.

The activity sheet concludes with a summary question (no. 18) that all students should be able to answer. In this sample, all students, no matter what level they complete, should be able to conjecture that a rectangle is the only parallelogram that can be inscribed in a circle. **Figure 6** shows student work that reflects different levels of complexity. Some students were able to justify their work only with a picture but lacked the precise mathematical language to write an explanation (see **figs. 6a** and **6b**). Other students were able to label diagrams with more information and write brief explanations (see **fig. 6c**), whereas some students were able to write a more formal proof. Because most students were able to illustrate the problem, we concluded the classes with a whole-group discussion in which all students could participate.

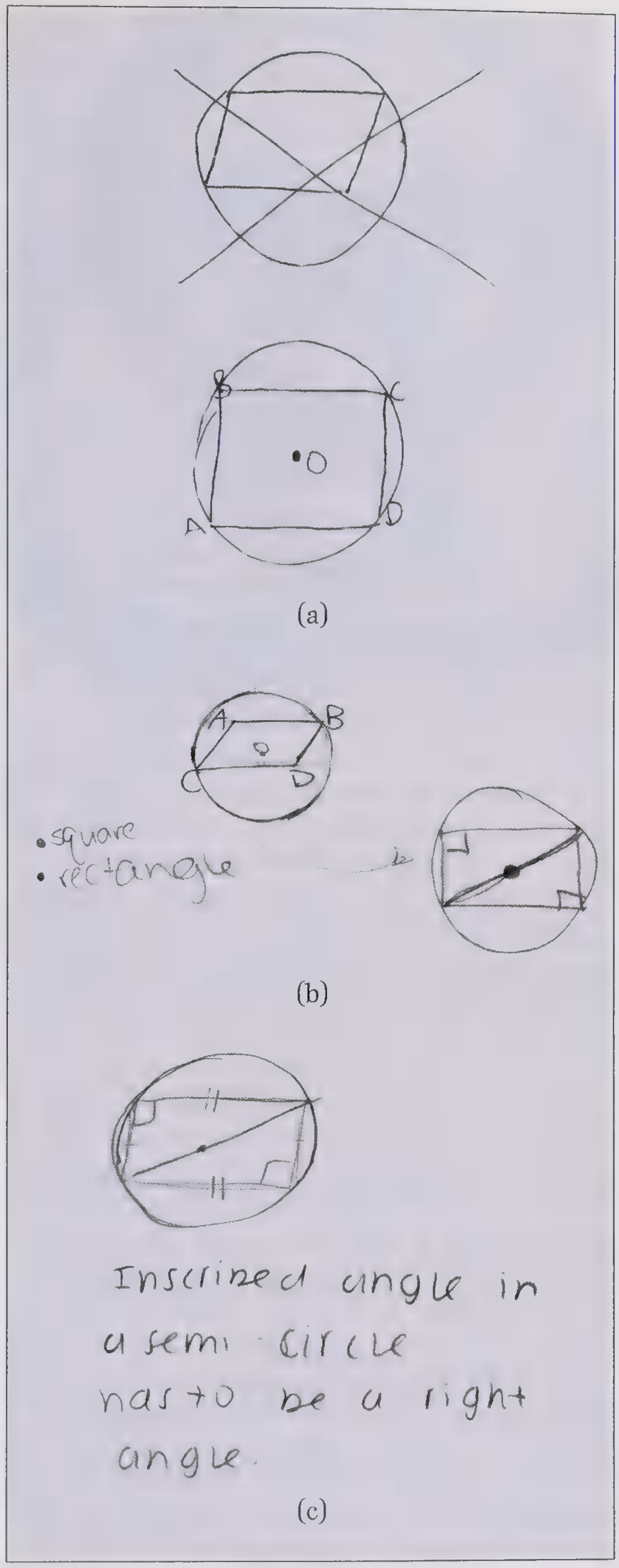


Fig. 6 These three examples of student work show differing levels of success.

CHALLENGES AND ADVANTAGES OF THE MODEL

While implementing this four-level model of parallel tasks in our classroom, we encountered several challenges. This model is not appropriate for every lesson. Some topics, such as introducing formal proofs, require a great deal of direct instruction that would be difficult to accomplish solely through parallel tasks. In addition, some students need help with selecting an appropriate level. We tested this model only for geometry, so we have not yet determined how it could be applied to other courses.

The greatest challenge with using this model is that planning effective lessons takes much more time and effort. We had to think much more carefully about what questions we asked. To determine each problem's level, we had to examine the type of

work required in the solution. In addition, we had to balance the amount of work for each level so that all students, no matter where they began, would be sufficiently challenged in class. If some levels required much less time to finish than others, then some students would finish early, whereas others would be "stuck" at their level. Fortunately, we did not have to create all the problems from scratch. For many topics, we were able to use problems from the textbook; we simply organized these by level.

Another challenge is assessing student work. We assessed classwork informally to give students the freedom to answer questions from different levels. During class, we circulated around the room to monitor progress and help students when necessary. Simply checking the number of questions completed did not accurately tell us what students understood; they could have been confused or discouraged by one question. However, talking to each student individually helped us determine what problems the student had with the material. Over time, these informal conversations enabled us to see trends in each student's work but required a great deal of class time. Teachers with limited class time may need to devise other ways to assess student work.

Although this model requires a great deal of effort to implement, we believe that it is worth the investment. It has helped us communicate our expectations more clearly to both students and parents. Although we have not done a formal study of this model, informal conversations with students and parents indicate that they appreciate knowing what specific work needed to be done for improvement. We were able to give more structured and specific feedback to students and parents about what students knew about a particular topic. And by labeling each problem's level of difficulty, we helped avoid giving too much work that tests only low levels of understanding or too much work that lacks proper development. Organizing problems in this way was particularly helpful for student teachers, who often struggled with creating work that had an appropriate level of difficulty.

This model of parallel tasks can improve student reasoning because it clearly shows what is required to achieve mastery. By allowing students to choose appropriate levels of work every day, we empower them to take more control of their learning. Creating an effective lesson using parallel tasks takes a great deal of time and effort. However, using the parallel tasks model—even for only a few lessons—can be a valuable experience for both teachers and students.

BIBLIOGRAPHY

Artzt, Alice F., Eleanor Armour-Thomas, and Frances R. Curcio. 2008. *Becoming a Reflective Mathematics Teacher: A Guide for Observations and Self-Assessment.* 2nd ed. New York: Taylor and Francis.

Burger, William F., and J. Michael Shaughnessy. 1986. "Characterizing the van Hiele Levels of Development in Geometry." *Journal for Research in Mathematics Education* 17 (1): 31–48.

Cirillo, Michelle. 2009. "Ten Things to Consider When Teaching Proof." *Mathematics Teacher* 103 (4): 251–57.

Clements, Douglas H. 2003. "Teaching and Learning Geometry." In *A Research Companion to Principles and Standards for School Mathematics,* edited by Jeremy Kilpatrick, W. Gary Martin, and Deborah Schifter, pp. 151–78. Reston, VA: National Council of Teachers of Mathematics.

National Council of Teachers of Mathematics (NCTM). 2000. *Principles and Standards for School Mathematics.* Reston, VA: NCTM.

Pierce, Rebecca L., and Cheryll M. Adams. 2005. "Using Tiered Lessons in Mathematics." *Mathematics Teaching in the Middle School* 11 (3): 144–49.

Small, Marian, and Amy Lin. 2010. *More Good Questions: Great Ways to Differentiate Secondary Mathematics Instruction.* New York: Teachers College Press.

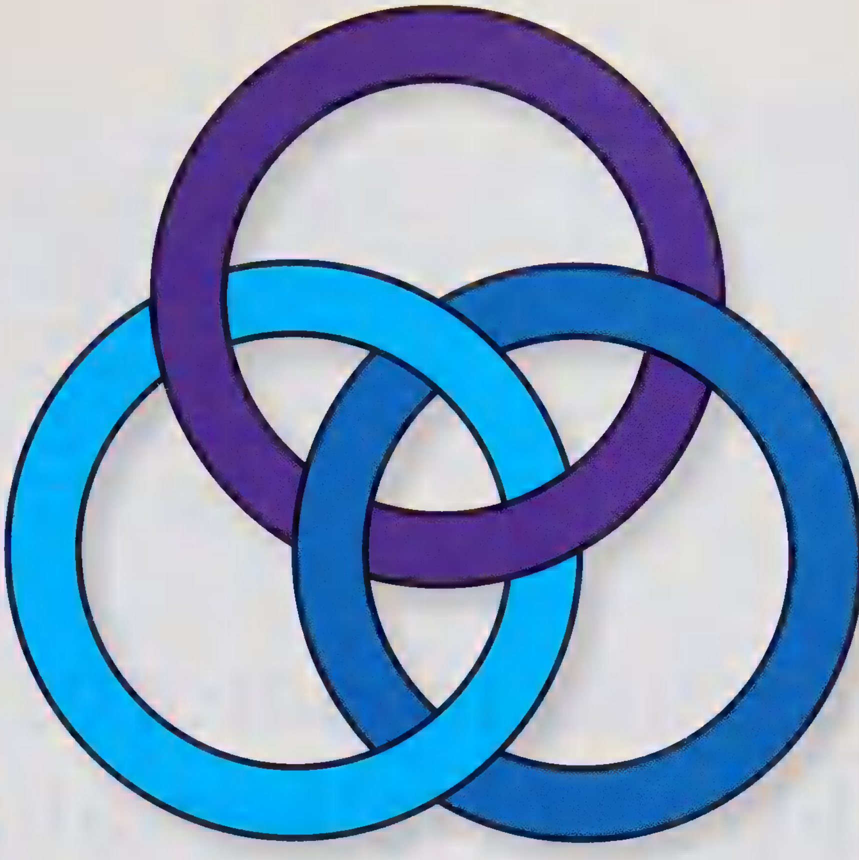
Useem, Elizabeth L. 1990. "You're Good, but You're Not Good Enough: Tracking Students out of Advanced Mathematics." *American Educator* 14 (3): 24–27, 43–46.

van Hiele, Pierre M. 1986. *Structure and Insight.* New York: Academic Press.

Webb, Norman L., Brian Vesperman, and Rob Ely. 2005. "Web Alignment Tool." Wisconsin Center of Educational Research, University of Wisconsin-Madison. http://wat.wceruw.org/index.aspx.

BOBSON WONG, bwong3@schools.nyc.gov, and **LARISA BUKALOV,** Lbukalo@schools.nyc.gov, teach mathematics at Bayside High School, a public high school in New York City. They are two-time recipients of the Math for America Master Teacher Fellowship.

CALL FOR MANUSCRIPTS



2015 Focus Issue

Creating Classroom Communities

NCTM guides mathematics teachers toward equitable teaching by emphasizing the importance of classroom communities. The Editorial Panel of *Mathematics Teacher* invites teachers, teacher educators, and education researchers to share their experiences in building classroom communities. We encourage submissions that will help *MT* readers learn new ways to capitalize on the strengths that cultural, racial, and linguistic diversity bring to our classrooms and schools.

Classroom communities embrace individuals.

- How can we foster a sense of belonging in our classrooms?
- How can we learn about our students—their interests, issues that might be important to them, their languages, and their racial or ethnic communities?
- How can we incorporate this knowledge in our lessons and assessments? What are examples of effective tasks that highlight strengths of individual students? How can we balance individual, cooperative, and whole-class activities?

Classroom communities foster communication.

- How do we organize lessons so that students can share their mathematical ideas or solutions?
- What classroom norms are effective in facilitating communication?
- How can we encourage students to listen to, critique, and build on other students' mathematical thinking?
- How can we communicate high standards for students?
- How can technologies foster student communication about mathematics?
- What strategies are successful in removing barriers to student participation and engagement in mathematics?

Please submit manuscripts at **mt.msubmit.net** by **May 1, 2014**. Be sure to enter the call's title (Creating Classroom Communities) in the Department/Calls field. No author identification should appear in the text of the manuscript. See **www.nctm.org/publications/content.aspx?id=22602** for manuscript guidelines. If you have ideas related to this topic and would like to discuss them before sending a manuscript, please contact Albert Goetz, agoetz@nctm.org.

NCTM NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

MATHEMATICS teacher

Thomas G. Edwards, S. Asli Özgün-Koca, and Kenneth R. Chelst

Controlling Inventory: Real-World Mathematical Modeling

ALEHDATS/ISTOCKPHOTO

Activities for Students appears six times each year in *Mathematics Teacher*, often providing in reproducible formats activity sheets that teachers can adapt for use in their own classroom. Manuscripts for the department should be submitted via http://mt.msubmit.net. For more information on the department and guidelines for submitting a manuscript, please visit http://www.nctm.org/publications/content.aspx?id=10440#activities.

Edited by ***Ruth Dover***
dover@imsa.edu
Illinois Mathematics and Science Academy
Aurora, IL

Patrick Harless
pdharless@gmail.com
University of Rochester (PhD student)
Rochester, NY

Amazon, Walmart, and other large-scale retailers owe their success partly to efficient inventory management. For such firms, holding too little inventory risks losing sales, whereas holding idle inventory wastes money. Therefore, profits hinge on the inventory level chosen. In this activity, students investigate a simplified inventory-control problem. Within this context, students develop tables, graphs, and algebraic representations to reach a decision. We have successfully completed this activity with students in both first- and second-year algebra.

Following the modeling cycle described by the Common Core State Standards (CCSSI 2010), this task—determining when and how many game consoles a store should order—promotes students' reasoning and sense making. The basic modeling cycle involves identifying variables and selecting those representing essential features of the problem; formulating a model by creating representations describing relationships between variables; analyzing and performing operations on those relationships to draw conclusions; interpreting results in terms of the original problem situation; validating the conclusions by comparing them with the situation; and then improving the model or, if it is an acceptable model, reporting on the conclusions and the reasoning behind them. Choices, assumptions, and approximations will always be part of this cycle.

Two questions related to inventory control are how many of the items to order and how often to order them. The unit of time might be one year, one quarter, or even a smaller time period. In the activity presented here, the business wants to minimize the total cost of carrying its inventory for a six-month period. The total cost to be minimized includes the ordering cost as well as the holding cost. Here holding costs include the money required to stock products—the cost of the storage space and the related labor—as well as the financial cost of foregone investment opportunities. The ordering cost includes any costs above the cost of the product ordered, such as delivery charges or the cost of labor required to place, receive, and stock an order.

The simplest inventory model that applies to problems like this is the basic Economic Order Quantity (EOQ). EOQ, originally developed at Westinghouse by Ford W. Harris in 1915, makes the following assumptions: The product will be ordered only when the current inventory has been exhausted; there is a fixed ordering cost that is independent of the number of items ordered; the product will be delivered instantaneously; and demand for the product is uniform over some period of time. In these days of overnight delivery, the third assumption is no longer far-fetched. Despite their simplicity, EOQ models have been and continue to be widely used. Moreover, the basic EOQ model provides a good starting point for understanding more complex models, such as those described in *Practical Management Science* (Winston and Albright 1997).

TEACHER NOTES

We recommend that teachers conduct a whole-class discussion to explain the problem context before implementing the activity. Questions may include these:

- When shopping for a particular item at a store, have you ever been unable to find it on the shelf? Did you ask an employee to check to see if the store had more of this item?
- Why do you think stores run out of products? Why wouldn't stores just keep a large supply of products in inventory?
- A store may place a few large orders or frequent small orders. What are the advantages and disadvantages of each option?

Alternatively, students could discuss these questions in small groups before coming together as a whole class to share their answers and the thinking behind them.

Following the discussion, students should understand the problem and be familiar with the terms *demand*, *order quantity*, and *order frequency*. *Demand* is the amount of product the company predicts that it can sell during a set time period. *Order quantity* is the amount of product ordered each time that an order is placed, and *order frequency* is the number of orders placed during the time period. Students should also know the meaning of *ordering cost*, *holding cost*, and *total cost*. Students should understand that in this scenario orders will be placed whenever inventory drops to zero and for the same amount as specified by the order quantity.

The table is designed to prompt students to develop algebraic representations. Developing these representations is crucial; without such representations of the model, students would be unable to comprehend the graphs and the spreadsheet that follow. Note that we chose to list only order quantities that are divisors of the demand quantity (350) to avoid, at the outset, the question of nonintegral order frequencies. To address the question of nonintegral order frequency, a teacher may ask about ordering 20 game consoles at a time. In this case, the result is 17.5 orders. This result represents a long-term average, which does not need to be an integer.

Initially, students may have difficulty calculating the total ordering cost for each order quantity in the table. Prompting students to think about the average size of the order (see column 4 in the table) may help. Students may also have difficulty determining the average number of game consoles in stock. In this case, the following questions help:

- When a new order is placed, how many game consoles would be in stock?
- When a new order arrives and is placed in stock, how many consoles would then be in stock?
- Would there ever be more than that number in stock?

To complete questions 4 and 5, we distributed a class set of TI-Nspire calculators with the spreadsheet and graphs preloaded, a step that greatly facilitated student exploration. The TNS files for the spreadsheet and graphs are available online, as are an Excel spreadsheet and TI-83/4 lists (go to **www.nctm.org/mt**).

In question 5, students should observe that the point on the total cost curve that represents the minimum total cost appears to coincide with the intersection of the ordering cost curve and the holding cost curve. In our classes, students made this connection readily. This result can be derived formally using calculus or verified by using a TI-device to compare the intersection with the minimum of the total cost curve.

After students have determined the optimal order quantity, they may explore modest deviations from the original assumptions. For example, if the game consoles must be ordered in specified lot sizes (e.g., five per box), it would no longer be feasible to choose the optimal order quantity exactly. Nevertheless, because of the curvature of the total cost function, small deviations from the optimal order quantity have little effect on the total cost. Students may explore this phenomenon by observing the graph, a table of values, or both.

Although we have presented the activity in one fifty-five-minute period, we recommend two class periods to allow enough time for class discussion and thorough exploration of questions 4 and 5 with technology.

CONCLUDING THOUGHTS

NCTM (2009) states that "a high school mathematics program based on reasoning and sense making will prepare students for citizenship, for the workplace, and for further study" (p. 3). They envision students "seeking patterns and relationships . . . , applying previously learned concepts to new problem situations, adapting and extending as necessary . . . making logical deductions based on current progress, verifying conjectures, and extending initial findings" (pp. 9–10). The Common Core State Standards for Mathematics modeling cycle echoes these sentiments. In this activity, students experience all these types of reasoning as they study linear and nonlinear relationships in an authentic context.

Students have responded favorably to this activity. Many commented that it was fun, enjoyable, or interesting. One wrote, "It made me think hard. It was challenging." Even students who at first found the activity difficult or unfamiliar made sense of it as they reasoned through it. When students experience

authentic mathematical modeling, they begin to view mathematics as a useful human endeavor with relevance to their lives. One first-year algebra student made this connection—the activity was "cool, because it [helped] me understand what I'm going to have to do when I own my own business."

REFERENCES

Common Core State Standards Initiative (CCSSI). 2010. Common Core State Standards for Mathematics. Washington, DC: National Governors Association Center for Best Practices and the Council of Chief State School Officers. http://www.corestandards.org/assets/CCSSI_Math%20Standards.pdf.

National Council of Teachers of Mathematics (NCTM). 2009. *Focus on High School Mathematics: Reasoning and Sense Making.* Reston, VA: NCTM.

Winston, Wayne L., and S. Christian Albright. 1997. *Practical Management Science: Spreadsheet Modeling and Applications.* Belmont, CA: Duxbury Press.

SOLUTIONS TO ACTIVITY

1. (*a*) Between March 1 and August 31, there are 184 days, counting Sundays. Thus, $350/184 \approx 1.9$; the average value need not be an integer. On average, over ten days Mark's father would sell two game platforms on nine days and one game platform on one day. Ask struggling students to consider the sales forecast and the total number of days.
   (*b*) 1 order of size 350; recommendations may vary.
   (*c*) 175 orders of size 2. If students answer 184, ask what the order quantity would be if they ordered 184 times. The resulting order quantity is fractional, which is impossible.
   (*d*) See the completed table.
   (*e*) The order cost and the order quantity vary inversely. That is, the cost decreases as the quantity increases; $\$150(350/Q)$.

2. (*a*) $(\$0.15)(184) = \$27.60$; $Q\ (\$0.15)(184)$.
   (*b*) As the number of game platforms held in stock increases, the holding cost increases. Similarly, the holding cost increases with the number of days that items are held in stock: $\$0.15QD$.
   (*c*) 5; the number in stock ranges

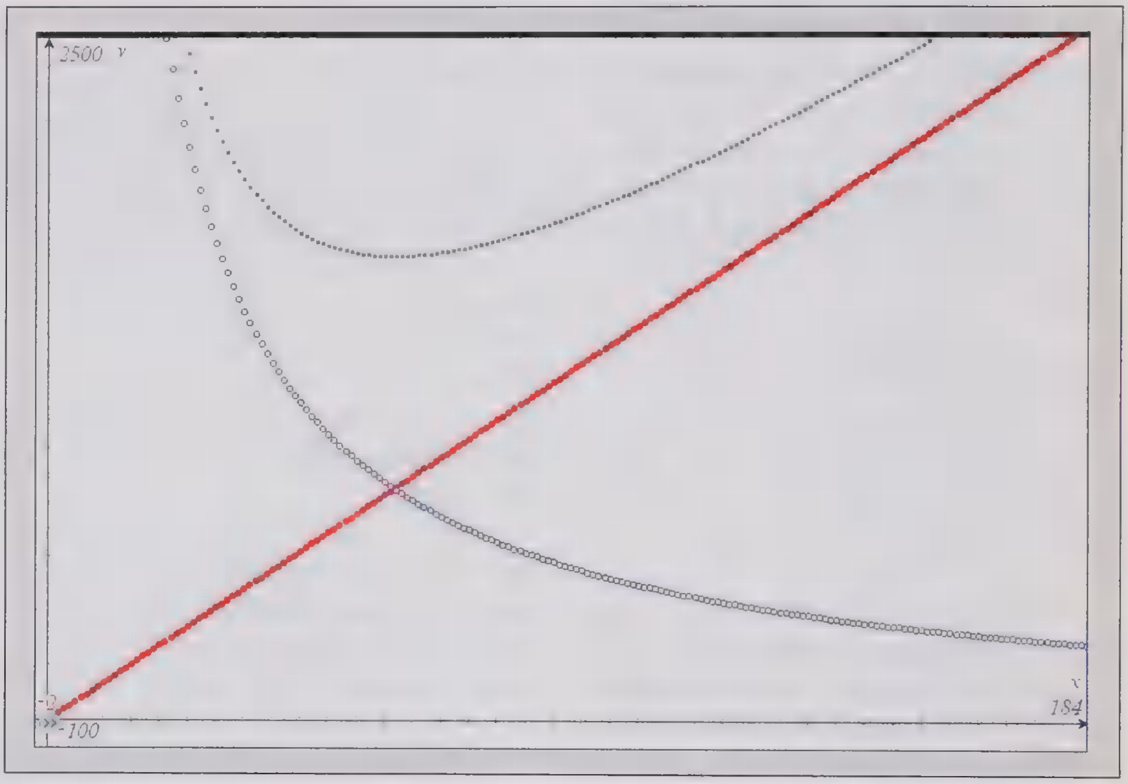


Fig. 1 The three graphs for ordering cost, holding cost, and total cost can be graphed on the same set of axes.

Table 1 Determining Inventory Costs

No. per Order	No. of Orders	Ordering Cost	Average No. in Stock	Cost/Day to Hold Average	Holding Cost for Mar. 1–Aug. 31	Total Cost
2	175	\$26,250	1	\$0.15	\$27.60	\$26277.60
10	35	\$5,250	5	\$0.75	\$138	\$5,388
14	25	\$3,750	7	\$1.05	\$193.20	\$3,943.20
25	14	\$2,100	12.5	\$1.875	\$345	\$2,445
35	10	\$1,500	17.5	\$2.625	\$483	\$1,983
50	7	\$1,050	25	\$3.75	\$690	\$1,740
70	5	\$750	35	\$5.25	\$966	\$1,716
175	2	\$300	87.5	\$13.125	\$2,415	\$2,715
350	1	\$150	175	\$26.25	\$4,830	\$4,980
Q	$350/Q$	$\$150 \cdot (350/Q)$	$Q/2$	$\$0.15 \cdot (Q/2)$	$\$0.15 \cdot (184) \cdot (Q/2)$	$\$150 \cdot (350/Q) + \$0.15 \cdot (184) \cdot (Q/2)$

from 0 to 10. If we assume that sales are uniform, the average is $0.5(0 + 10) = 5$.

(*d*) $\$0.15(5) = \$0.75$; $\$0.15(5)(184) = \$138$; the average holding cost per day is the product of holding cost per day and average number of items in stock. Multiply by 184 days for the total holding cost.

(*e*) See **table 1** for all answers.

3. (*a*) The total cost is the sum of the ordering and holding costs.
   (*b*) See **table 1**.
   (*c*) As the order quantity increases, total cost first decreases but then increases; $\$150(350/Q) + \$0.15(184)(Q/2)$.

4. (*a*) Probably 70; this shows the smallest total cost.
   (*b*) Answers will vary. If students consider only divisors of 350, remind them that the order quantity represents an average and need not be an integer.
   (*c*) The additional data show that total cost first decreases and then increases with the order quantity. They also reveal order quantities with smaller total costs.
   (*d*) Probably 62, since its total cost is smallest. This involves a non-integral solution of 5.6 orders per period.

5. (*a*) See **figure 1**.
   (*b*) Answers will vary. Holding cost increases with Q, whereas ordering cost decreases with Q. Total cost first decreases and then increases. The holding cost graph is linear, but the other two graphs are nonlinear.
   (*c*) About 60; this is where the total cost graph is very close to its minimum. The other two graphs intersect for the same value of the order quantity. A policy of 60 could be implemented with five orders of 60 and one order of 50.
   (*d*) They are all close. Five orders of 70 is the simplest policy, but five orders of 60 and one order of 50 is less costly.




THOMAS G. EDWARDS, t.g.edwards@wayne.edu, is an associate dean at Wayne State University in Detroit, Michigan. **S. ASLI ÖZGÜN-KOCA,** aokoca@wayne.edu, is a mathematics educator in the College of Education at Wayne State University. **KENNETH R. CHELST,** kchelst@wayne.edu, is an operations researcher in the College of Engineering at Wayne State University.

more4U

For TNS files for the spreadsheet and graphs as well as an Excel spreadsheet and TI-83/4 lists, go to **www.nctm.org/mt**.

HOW MANY TO ORDER?

Mark Steele's father owns a store that sells video game consoles, including the popular AllSport platform, which sells for $299.95. The store has limited storage space, however, so Mr. Steele must plan his orders carefully. He wants to minimize the total cost of ordering AllSport platforms and holding them in stock while still selling as many as possible. Mark thinks that he can help. Here is the information that Mark's father has provided.

The local distributor of AllSport platforms maintains a warehouse just outside the city and guarantees next-morning delivery of all order requests. Mr. Steele will place an order whenever he runs out and will receive the order the next morning.

Mr. Steele will order the same number of AllSport platforms each time he places an order.

The fixed cost of each order regardless of the size of the order, called the *order cost*, is $150. This is in addition to the price of each console. The order cost includes the value of Mr. Steele's time to order, receive, and stock the consoles as well as the cost of delivering them to the store.

Mr. Steele calculates his holding cost to be $0.15 per console per day. The holding cost includes the cost of storage as well as the financial cost of having money invested in inventory instead of earning money in another investment. For example, if 10 AllSport platforms were in storage one day, their holding cost for that day would be $1.50.

Mr. Steele forecasts selling a total of 350 AllSport platforms between March 1 and August 31. Figures from past years show that sales have been approximately evenly distributed over this time period.

1. First, Mark and his father need to consider the ordering cost. Assume that the store is open every day between March 1 and August 31.
   (*a*) On average, how many AllSport platforms does Mr. Steele forecast selling each day? How can you interpret this?
   (*b*) What is the minimum number of times Mr. Steele can order consoles to cover the time period between March 1 and August 31? Should Mark recommend this minimum strategy?
   (*c*) What is the maximum number of orders that Mr. Steele would need to place? Should Mark recommend this maximum strategy?
   (*d*) Determine the number of orders required and the ordering cost for each quantity listed in the table.
   (*e*) How does the order cost vary with the order quantity? Describe the relationship in words and write an algebraic expression that represents the relationship.

2. Mark also needs to consider the holding cost.
   (*a*) How much would it cost to hold one AllSport platform in stock for the entire period? How much would it cost to hold Q AllSport platforms for the entire period?
   (*b*) How is the holding cost related to the quantity held in stock? How is it related to the number of days that the items are held? Write an algebraic expression that represents the cost of holding Q consoles for D days.
   (*c*) Suppose that Mark and his father decide to order 10 AllSport platforms at a time. On average, how many consoles will be in stock? Explain your reasoning.
   (*d*) If we assume that Mr. Steele orders 10 consoles at a time, what will be the holding cost for a typical day? What will be the holding cost for the entire period? Again, explain your reasoning.
   (*e*) For each order quantity shown in the table, determine the average number of consoles held in stock, the average holding cost per day, and the expected total holding cost for the entire period. Write an algebraic expression that relates each of these to the order quantity, Q.

HOW MANY TO ORDER? (continued)

3. Finally, Mark needs to calculate the total cost.
   (*a*) Using the information that you have collected, explain how you can determine the total cost for a given order quantity.
   (*b*) Determine the total cost associated with each order quantity listed in the table.
   (*c*) How does the total cost vary with the order quantity? Describe the relationship in words and write an algebraic expression that represents the relationship.

Table 1 Determining Inventory Costs

No. per Order	No. of Orders	Ordering Cost	Average No. in Stock	Cost/Day to Hold Average	Holding Cost for Mar. 1–Aug. 31	Total Cost
2						
10						
14						
25						
35						
50						
70						
175						
350						
Q						

4. Mark and his father wish to minimize the total costs.
   (*a*) According to the table, which order quantity might Mark recommend? Why would this be a good choice?
   (*b*) Additional data may help Mark make his recommendation. What additional order quantities might Mark consider?
   (*c*) If you have a TI-Nspire, open the spreadsheet provided by your teacher. If not, work with a partner to compute the total cost for additional order quantities that Mark might recommend. What new observations can you make about the relationship between the order quantity and the total cost?
   (*d*) According to the spreadsheet, which order quantity might Mark recommend? Explain how you chose this quantity.

5. A graph may provide additional helpful information.
   (*a*) Plot graphs of the ordering cost, holding cost, and total cost on one set of axes.
   (*b*) Looking at the graphs, what do you observe? Describe the graphs as precisely as you can.
   (*c*) According to the graphs, which order quantity should Mark recommend? What do you notice about the cost curves at this quantity?
   (*d*) Compare the recommendations that you identified according to the original table, the spreadsheet, and the graph. Are they the same? On the basis of all the information, which order quantity should Mark recommend?

Craig J. Cullen, Joshua T. Hertel, and Sheryl John

Investigating Extrema with GeoGebra

Technology can be used to manipulate mathematical objects dynamically while also facilitating and testing mathematical conjectures. We view these types of authentic mathematical explorations as closely aligned to the work of mathematicians and a valuable component of our students' educational experience. This viewpoint is supported by NCTM and the Common Core State Standards for Mathematics (CCSSM).

NCTM's Technology Principle, for example, states: "Technology enriches the range and quality of investigations by providing a means of viewing mathematical ideas from multiple perspectives" (NCTM 2000, p. 25). Similarly, the first of CCSSM's eight mathematical practices—"Make sense of problems and persevere in solving them"—encourages teachers to work toward developing students who "check their answers to problems using a different method" (CCSSI 2010, p. 6). In the fifth mathematical practice—"Use appropriate tools strategically"—proficient mathematics students are described as understanding that "technology can enable them to visualize the results of varying assumptions, explore consequences, and compare predictions with data." (p. 7). In addition, CCSSM states that students should recognize the help that mathematical tools can offer as well as their limitations.

Here we discuss an exploration that we have used in the classroom with preservice secondary school teachers. We believe that it provides an excellent venue for students to experience the use of technology consistent with both NCTM's Technology Principle and CCSSM's mathematical practices. Specifically, it allows them to explore mathematics in multiple representations (algebraic and graphical) while experiencing both the benefits and the limitations of a powerful mathematical tool.

The exploration we present is based on an exploration suggested in NCTM's discussion of its Algebra Standard (NCTM 2000):

> . . . students should learn to recognize how the values of parameters shape the graphs of functions in a class. With access to computer algebra systems (CAS) . . . students can easily explore the effects of changes in parameter as a means of better understanding classes of functions. For example, explorations with functions of the form $y = ax^2 + bx + c$ lead to some interesting results. (p. 299)

The activity outlined in this excerpt—exploring how the graph of a quadratic equation (i.e., $y = ax^2 + bx + c$) changes as the parameters a, b, and c vary—is a rich and interesting investigation that has been discussed in earlier *Mathematics Teacher* articles (Edwards and Özgün-Koca 2009; Fallon and Luck 2010). We extend this exploration by investigating higher-degree polynomials. Although our work is with preservice secondary school teachers, we believe that this activity is appropriate for high school students in precalculus and calculus. Further, components may be suitable for students in an advanced algebra course. In the interest of thoroughness, we will work through the activity as a calculus task.

PROBLEM SETUP

To extend the original investigation to higher-degree polynomials, we must address an immediate question: Which point or points on the polynomial should we track as a parameter changes? The original task concerns tracking the location of the vertex of a quadratic. In the case of a quadratic, the location of the vertex is a unique point on the graph that represents a *local extremum* (i.e., a maximum or a minimum), the coordinates of which can be found by setting the first derivative equal to zero and solving.

If we extend this idea to higher-degree polynomials, finding the first derivate, setting it equal to zero, and solving will again help us find local maximums or minimums. However, unlike with the quadratic, these extrema will not necessarily be unique points (or, for that matter, may not exist at all). Thus,

Technology Tips, which provides a forum for innovative uses of technology in the teaching and learning of mathematics, appears seven times each year in *Mathematics Teacher*. Manuscripts for the department should be submitted via http://mt.msubmit.net. For more background information on the department and guidelines for submission, visit http://www.nctm.org/publications/content.aspx?id=10440#tech.

Edited by ***Larry Ottman***
lottman@gfsnet.org
Germantown Friends School
Philadelphia, PA

James Kett
j.gkett@gmail.com
Singapore-American High School (retired)
Singapore

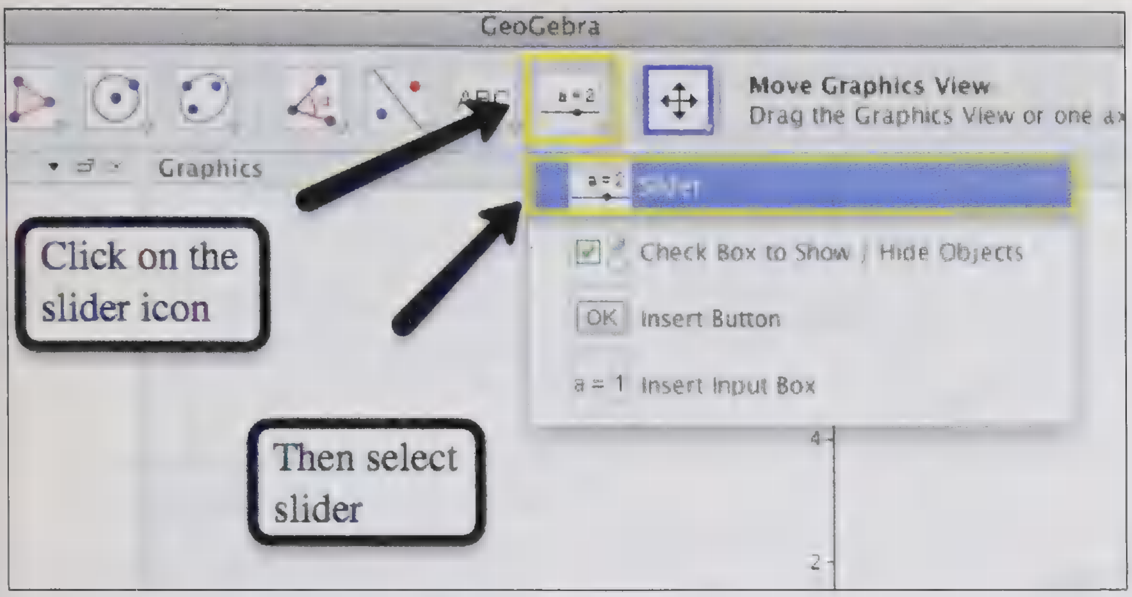


Fig. 1 Students click on the icon to create a slider.

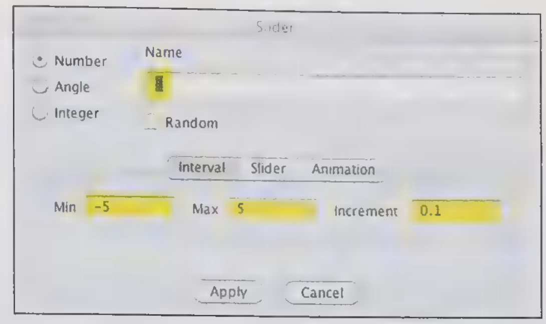


Fig. 2 Students set their slider inputs.

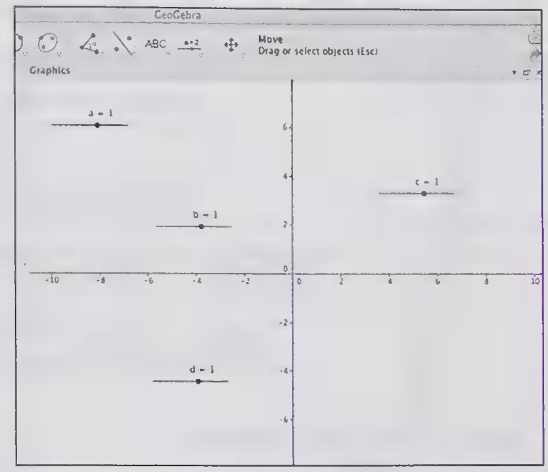


Fig. 3 The four sliders are displayed on the page.

simply answering the question can be a challenge. Because our aim is to engage students in mathematical exploration so that they might learn how to use technology to develop and test conjectures, we shall narrow our focus to the case of a cubic in standard form and consider the path traced by the extrema as we vary the parameters individually.

INVESTIGATING THE PATH OF EXTREMA

Using the dynamic geometry software GeoGebra (available for free from www.geogebra.org), we can easily construct a dynamic graph of any polynomial—for example, a cubic function in standard form, $y = ax^3 + bx^2 + cx + d$, where a is nonzero and where a, b, c, and d are real numbers. To do so, we will use the built-in slider tool. First, we click on the slider icon and then click on **Slider** from the drop-down menu (see **fig. 1**). We can now place a slider on the page by simply clicking on the page. Once we click on the page, a slider box will open up (see **fig. 2**), prompting us to input settings for the slider to be created. These settings include a name, the interval under consideration, and the increment. We have entered a for the name and left the default settings of **Min -5** and **Max 5** and increment **0.1**. Note that these settings can be adjusted later. After entering the settings, we click on **Apply** in the slider box, leaving us with a slider labeled a on our page. Repeating the process described, we create sliders for b, c, and d (see **fig. 3**).

Note that the sliders are located on the page where we clicked. To move them, we first click on the selection icon in the upper-left corner. Next, hovering over one of the sliders and right-clicking will open up a menu showing the setting for the slider. Unchecking **Fix Object** will allow the slider to move when we click and hold the line segment (see **fig. 4**).

Now we enter the function that we want to explore. To do this, we type `a*x^3+b*x^2+c*x+d` into the input bar at the bottom of the screen and press **Enter** (see **fig. 5**). Note that if we type the expression as shown here, GeoGebra will automatically assign a function name. An appropriate function name, such as $h(x)$, can also be specified by students. The resulting graph will look similar to that shown in **figure 6**.

The name of this function will be defined in the algebra window on the left of the screen. In a new sketch, the function should automatically be named f. The parameters of the cubic can now be changed by manipulating the sliders with the mouse or by entering `parameter name=value` into the input bar. For example, to set the value of parameter b to 4, we would type `b=4` into the input bar.

The extrema for the function can be constructed by using the built-in **Extremum** command. To construct the extrema of the cubic function f, enter `Extremum[f]` into the input bar. Note that in **figure 1**, $a = b = c = d = 1$, and there are no extrema. A document with $b = 4$ should have two extrema. Students and teachers can experiment and observe how changing the value of each parameter changes the location of the extrema.

We are now ready to consider the primary focus of our exploration: What is the path of the extrema when each of the parameters a, b, c, and d is manipulated individually? Although students can develop conjectures using only observation, we have found it helpful to use the built-in trace feature to visualize the path. To trace a point, we right-click on the point and then left-click on the **Trace On** option. Once we have turned on the trace for the extrema, changing the value of a parameter results in a static representation of the path of the extrema displayed as a collection of plotted points. To erase the collection of traces, we right-click on either of the points and uncheck **Trace On**. **Figure 7** shows the trace when a is varied.

After experimenting with all the parameters for a few minutes, we can generate several conjectures about the path of the extrema. A few examples follow:

- When parameter a is changed, the extrema appear to follow a quadratic path.
- When parameter b is changed, the extrema appear to follow a cubic path.
- When parameter c is changed, the extrema appear to follow a cubic path.
- When parameter d is changed, the extrema appear to follow a vertical path.

Although GeoGebra is useful for generating and testing conjectures, it is not helpful for generating the equations that model these paths. We will show how these equations can be generated using algebra and basic calculus concepts.

PROVING CONJECTURES

Let's first consider finding the path created by the extrema when parameter a is changed. As previously mentioned, the value of the derivative of f at each of the extrema points is zero. Therefore, we can use the derivative to find an equation that relates a and x. Using this information, we can then find an equation that models the path of the extrema as a is changed. Beginning with the general form of the cubic, $f(x) = ax^3 + bx^2 + cx + d$, we set the first derivative $f'(x) = 3ax^2 + 2bx + c$ equal to zero and solve for a:

$$a = \frac{-2bx - c}{3x^2}$$

and

$$f(a) = f\left(\frac{-2bx - c}{3x^2}\right) = \frac{1}{3}\left(bx^2 + 2cx\right) + d.$$

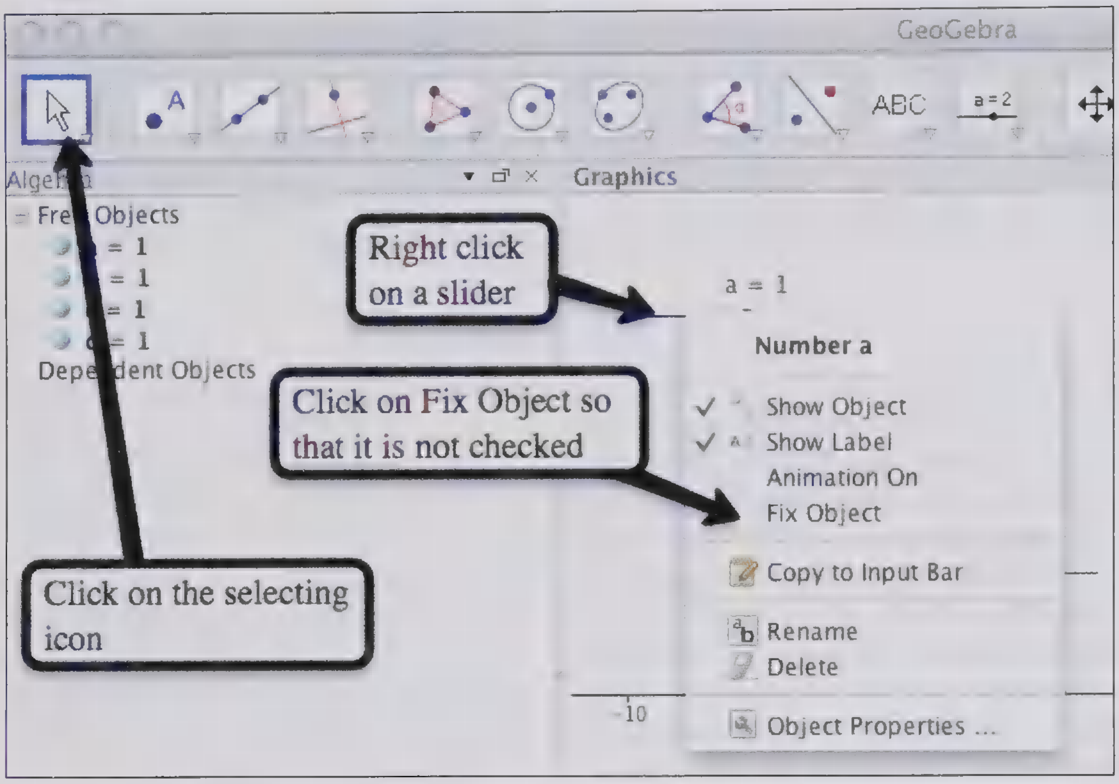


Fig. 4 Now the sliders are more properly aligned on the page.

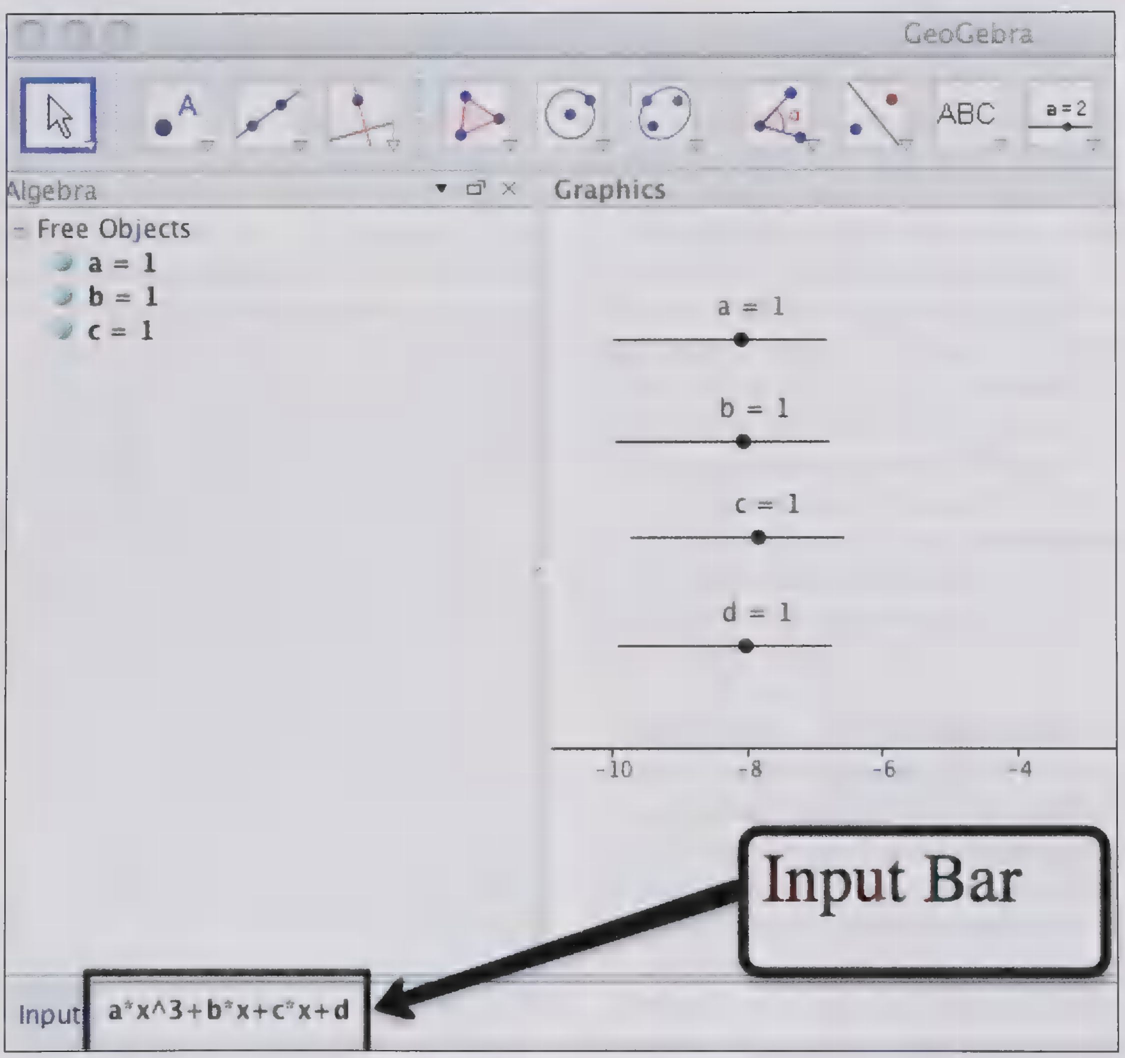


Fig. 5 The input bar is used to enter the cubic function into GeoGebra.

The function f models the path followed by the extrema. Students can verify this by plotting f in GeoGebra and comparing it with the path created by the trace of the extrema. We enter the function above into the input bar in GeoGebra and then vary a to verify that the extrema do in fact fall along the path defined by $y = 1/3(bx^2 + 2cx) + d$ (see **fig. 8**). Using the same approach, we can find and verify the equations for the paths followed by the extrema when b and c are changed.

The equation for the path followed by the extrema when d is changed cannot be found using this method because $f'(x)$ does not depend on d. Instead, we need to shift our attention to the ordered pairs (x, y) that define the extrema. To find the ordered pair mentioned above, we set $f'(x) = 0$, solve for x, and then substitute this x-value into $f(x)$ to find the corresponding y-value. The resulting ordered pair has x-coordinate

$$\left(\frac{-b \pm \sqrt{b^2 - 3ac}}{3a}\right)$$

and y-coordinate

$$\left(a\left(\frac{-b \pm \sqrt{b^2 - 3ac}}{3a}\right) + b\left(\frac{-b \pm \sqrt{b^2 - 3ac}}{3a}\right) + c\left(\frac{-b \pm \sqrt{b^2 - 3ac}}{3a}\right) + d\right).$$

By focusing our attention on the fact that the expression for x does not

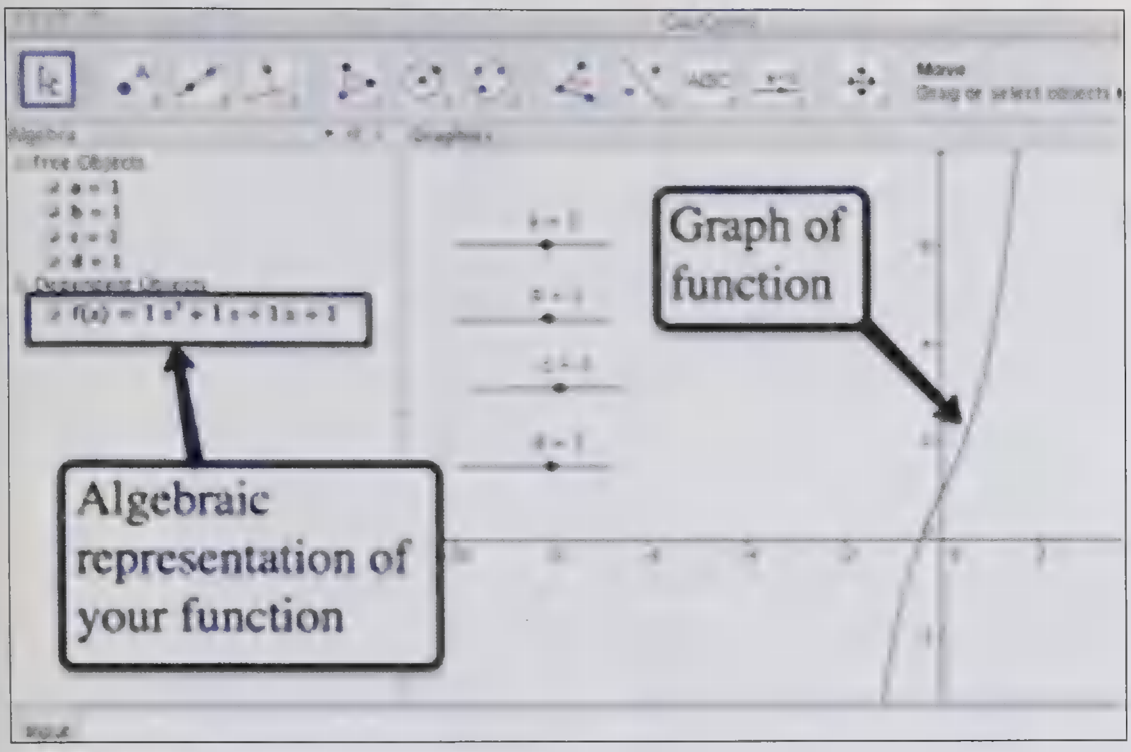


Fig. 6 The graph is now ready to be manipulated.

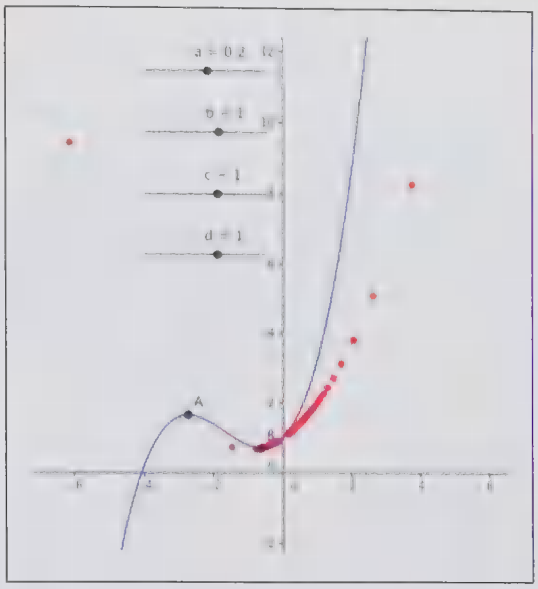


Fig. 7 This screen shot shows the trace extrema of the curve $y = ax^3 + x^2 + x + 1$ at points A and B as parameter a is varied.

depend on d whereas the expression for y does, we can deduce that a change in the value of d will result in the extrema tracing the vertical line from the x-coordinate above. In other words, changing the parameter d does not change the x-coordinate of the extrema but does change the y-coordinate.

A more rigorous mathematical method to determine the path traced by the extrema when d varies is to analyze the effects of a change in parameter d on the x- and y-values of the extrema individually. Essentially, we notice that the derivative of x with respect to d is 0, whereas the derivative of y with respect to d is 1:

Thus, it can be shown that

$$\frac{\left(\frac{dy}{dd}\right)}{\left(\frac{dx}{dd}\right)} = \frac{dy}{dd} \circ \frac{dd}{dx} = \frac{dy}{dx} = \frac{1}{0}.$$

This illustrates that as d is changed, the extrema will trace a path that has an undefined slope—that is, a vertical line.

A REALISTIC MATHEMATICAL EXPERIENCE

Throughout this investigation, we see opportunities for students to engage in three important mathematical activities: generating, testing, and proving mathematical conjectures. The use of technology—in this case, GeoGebra—allows students to observe a phenomenon as it occurs. After clearly stating this observation and testing it under multiple conditions, students are typically convinced that their conjecture is true. With this "certainty" in mind, they are motivated to find the mathematical model, necessitating a shift away from the technology that they used to generate the conjecture.

Through this process, mathematically proficient students can engage in at least two of CCSSM's eight mathematical practices. First (and most directly), students used tools strategically (practice 5) as they tried to make sense of problems and persevere in solving them (practice 1). Second, students had the opportunity to experience the benefits of the mathematical tool they used—in this case, GeoGebra—to explore and visualize while realizing the limitation of the tool when it came time to generate a specific rule for the path traced by the extrema. This process allows students to engage in a more realistic mathematical experience, including the discovery and generation of a conjecture and the resulting mathematical analysis.

The exploration described here, used in a technology-rich environment, provides an excellent opportunity for students to engage in a realistic mathematical experience. Throughout the experience, students work through all aspects of the proof process: exploration, conjecture, and proof (Burke et al. 2008). Although this particular exploration focused on the paths of the extrema of a cubic, it can be extended to higher-degree polynomials and other interpretations (e.g., tracking the extrema for the general quartic).

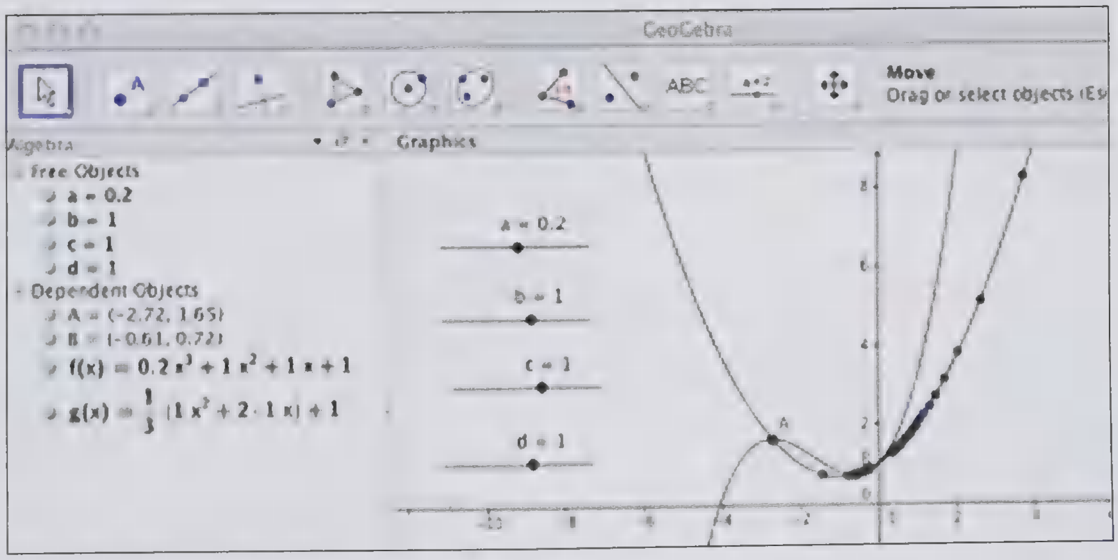


Fig. 8 This screen shot displays the graph of the original function, the collection of traced points, and the predicted path of the traced points.

On the basis of our initial work, we have generated several conjectures about the behavior of the extrema for any polynomial of degree n:

- For any polynomial of degree n with leading coefficient a_n, the path of the extrema when a_n is varied will be a polynomial of degree $n - 1$.
- For any polynomial of degree n with constant term a_0, the path of the extrema when a_0 is varied will be a vertical line.
- For any polynomial of degree n with coefficients $a_n, a_{n-1}, \ldots, a_0$, the path of the extrema when any one of $a_{n-1}, \ldots, a_1$ is varied will be a polynomial of degree n.

Proving (or disproving) these conjectures could be meaningful extensions of this investigation. We encourage educators and students to explore this activity and consider the potential it holds for bringing meaningful mathematics into the secondary school classroom.

REFERENCES

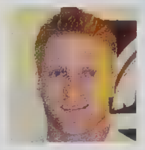
Burke, Maurice J., Jennifer Luebeck, Tami S. Martin, Sharon M. McCrone, Anthony V. Piccolina, and Kate J. Riley. 2008 *Navigating through Reasoning and Proof in Grades 9–12.* Reston, VA: National Council of Teachers of Mathematics.

Edwards, Thomas G., and Asli Özgün-Koca. 2009. "Creating a Mathematical 'B' Movie: The Effect of *b* on the Graph of a Quadratic." *Mathematics Teacher* 103 (3): 214–20.

Fallon, Danielle, and Gary S. Luck. 2010. "Exploring the ABCs of Parabolas." *Mathematics Teacher* 104 (2): 214–20.

Common Core State Standards Initiative (CCSSI). 2010. Common Core State Standards for Mathematics. Washington, DC: National Governors Association Center for Best Practices and the Council of Chief State School Officers. http://www.corestandards.org/assets/CCSSI_Math%20Standards.pdf.

National Council of Teachers of Mathematics (NCTM). 2000. *Principles and Standards for School Mathematics.* Reston, VA: NCTM.



CRAIG J. CULLEN, cjculle@ilstu.edu, a former high school mathematics teacher, teaches mathematics education courses at Illinois State University in Normal. He is interested in children's development of measurement concepts and the role of technology in mathematics education. **JOSHUA T. HERTEL,** herteljt@gmail.com, is an assistant professor in the mathematics department at the University of Wisconsin–La Crosse. A former high school mathematics teacher, he is interested in the teaching and learning of trigonometry, the role of technology in mathematics education, and the development of probabilistic reasoning. **SHERYL JOHN,** sheryl.john@icc.edu, is an adjunct developmental mathematics instructor and the math lab coordinator at Illinois Central College in East Peoria. She is interested in the use of technology for problem-solving skills through the enhancement of critical thinking and comprehension of mathematical concepts.

Richard E. Auer and Michael P. Knapp

Studying Baseball's Wild-Card Team Using Probability

The modern era of professional baseball playoffs began in 1903, when the champions of the American League and the National League played the first World Series. Except for one year, 1904, this playoff system was maintained until 1969. Beginning in 1969, each of the two leagues in Major League Baseball (MLB) was divided into two divisions to accommodate the addition of extra teams in each league. From then through 1993, the two divisional champions in each league played against each other in an initial playoff round to determine which teams would go on to the World Series.

In 1994, as the result of more expansion, each league was reorganized into three divisions. To maintain a tidy playoff system, each league began sending a "wild card" team into the first of three rounds of playoffs. This wild-card team would simply be the best of the three second-place finishers and would join the three divisional winners in the first round of games.

Ever since MLB began discussing the use of wild-card teams, controversy has ensued. Hall of Famer Johnny Bench was quoted as saying, "Someday, we'll have a team that wins the World Series without winning their own division" (Lawes 1992, p. 3). As recently as 2006, Pulitzer Prize nominee Rick Hummel complained, "Already baseball has been burned (some would say scarred) by the fact that three of the past four World Series winners have been wild card teams. . . . Baseball needs to make it much harder for a wild card team" (Hummell 2006, p. C6).

To understand the three-division playoff system, consider the final 2006 American League standings. The divisional champs—the New York Yankees, the Minnesota Twins, and the Oakland Athletics—advanced. Among the second-place teams, the Detroit Tigers had the highest winning percentage, even higher than Oakland's, and hence were labeled the wild-card team. In the playoffs, the wild-card Tigers defeated the Yankees and Athletics to advance to the World Series, where they lost to a division-winning St. Louis Cardinals team from the National League. The Cardinals had only the fifth-best record in the National League, but as winners of their division they advanced to the playoffs over two teams with better records.

The purpose of this article is to use basic probability to support the idea of having wild-card teams in MLB playoffs. We will develop a model for the years 1994 through 2011, when the American League had 14 teams, the National League had 16 teams, and each league sent one wild-card team to the playoffs. Using this model, we will show that the wild-card team should probabilistically be better than at least one of the division winners, just as in 2006.

THE PROBABILISTIC MODEL

Our model is simple. We will explain it using the 2006 American League results and then state those for the National League. To begin, we replace the team names with their rankings within the league.

Delving Deeper offers a forum for classroom teachers to share the mathematics from their own work with the journal's readership; it appears in every issue of *Mathematics Teacher*. Manuscripts for the department should be submitted via http://mt.msubmit.net. For more background information on the department and guidelines for submitting a manuscript, visit http://www.nctm.org/publications/content.aspx?id=10440#delving.

Edited by ***Maurice Burke****, Maurice.Burke@utsa.edu*
University of Texas at San Antonio

J. Kevin Colligan*, jkcolligan@verizon.net*
RABA Center of SRA International, Columbia, MD

Maria Fung*, mfung@worcester.edu*
Worcester State College, Worcester, MA

Jeffrey J. Wanko*, wankojj@muohio.edu*
Miami University, Oxford, OH

For the American League, the teams are ranked 1 through 14. Within this ranking system, the New York Yankees would be team 1 because they had the best record in the league in 2006. Minnesota would then be team 2, and so on.

The fundamental assumption that we make in our model is that the rankings of the 14 teams for any given season are randomly distributed across the three divisions. Assume that we have a set of 14 balls with 5 inscribed East, 5 inscribed Central, and 4 inscribed West. To randomly assign a division to team 1, the best team, we choose at random a ball from the 14. We continue this process for teams 2 through 14, without replacing the chosen balls.

Our probabilistic model considers two random variables: (1) the rank of the wild-card team, denoted WC; and (2) the rank of the worst divisional winner, denoted WDW. We will show that the expected value of the rank of the wild-card team, $E(WC)$, is smaller than the expected value of the rank of the worst divisional winner, $E(WDW)$. That is, the wild-card team is expected to be better than the worst divisional winner. In fact, the model suggests that, in more than 5% of seasons, the worst divisional winner will rank in the bottom half of the league.

Because there are three divisions, it is not hard to see that the wild-card team (the best of the second-place teams) will always have rank 2, 3, or 4. Similarly, the worst divisional winner can have only rank 3 through 11. To calculate the expected rank of the wild-card team, we need to apply the notion of a discrete expected value:

$$E(WC) = 2 \cdot P(WC = 2) + 3 \cdot P(WC = 3) + 4 \cdot P(WC = 4),$$

where $P(X)$ represents the probability that some event X occurs.

Let's think first about $P(WC = 2)$. The only way to have $WC = 2$ is for team 1 and team 2 to be in the same division. We will use the multiplication rule for probability:

$$\begin{aligned} P(WC = 2) = {} & P(\text{team 2 is in East given that team 1 is in East}) \cdot P(\text{team 1 is in East}) \\ & + P(\text{team 2 is in Central given that team 1 is in Central}) \cdot P(\text{team 1 is in Central}) \\ & + P(\text{team 2 is in West given that team 1 is in West}) \cdot P(\text{team 1 is in West}) \end{aligned}$$

We can easily see that P(team 1 is in East) = 5/14 because there are 5 teams in the East and none of the 14 teams' rankings is yet assigned. Then P(team 2 is in East given that team 1 is in East) = 4/13 because there are only 4 East division slots left and 13 remaining teams. In this way,

$$P(WC = 2) = (4/13)(5/14) + (4/13)(5/14) + (3/13)(4/14) \approx .2857.$$

Already, we have a very interesting result. Under this model, for more than 28% of the time, the wild-card team will be the second-best team in the American League.

Now consider $P(WC = 3)$. To have $WC = 3$, two conditions are necessary: (1) team 1 and team 2 must be in different divisions; and (2) team 3 must be in the same division as one of those two. To calculate all the possible combinations, let $Q(X, Y)$ represent the probability that team 1 is in division X, team 2 is in division Y, and team 3 is in either division, X or Y. If E, C, and W are the possible outcomes of X and Y, then

$$P(WC = 3) = Q(E, C) + Q(C, E) + Q(E, W) + Q(W, E) + Q(C, W) + Q(W, C).$$

Now we have

$$\begin{aligned} Q(E, C) = {} & P(\text{team 3 is in East or Central given that team 1 is in East and team 2 is in Central}) \\ & \cdot P(\text{team 2 is in Central given that team 1 is in East}) \cdot P(\text{team 1 is in East}). \end{aligned}$$

Using reasoning similar to that used earlier, we can see that $Q(E, C)$ is (8/12)(5/13)(5/14) and that

$$\begin{aligned} P(WC = 3) = {} & (8/12)(5/13)(5/14) \\ & + (8/12)(5/13)(5/14) + (7/12)(4/13)(5/14) \\ & + (7/12)(5/13)(4/14) + (7/12)(4/13)(5/14) \\ & + (7/12)(5/13)(4/14) \\ & \approx .4396. \end{aligned}$$

Although we will not show the details, we can use similar reasoning to show that $P(WC = 4) \approx .2747$. We now see that

$$E(WC) \approx 2(.2857) + 3(.4396) + 4(.2747) \approx 2.989.$$

Because this expected rank is less than 3 and the worst divisional winner must rank at least 3, our model suggests that the wild-card team is, on average, better than the worst divisional winner.

It is also possible to find the probability that the wild-card team *is* better than the worst divisional winner. Because this happens exactly when the wild-card team has rank 2 or 3,

$$P(WC < WDW) = P(WC = 2) + P(WC = 3) \approx .2857 + .4396 \approx .7253.$$

So the model suggests that the wild-card team will be better than the worst divisional winner about 72.53% of the time.

THE WORST DIVISIONAL WINNER

Now, how good can we expect the worst divisional winner to be? Clearly, the best possible ranking that the worst divisional winner can attain is 3. At the opposite extreme, if the four worst teams all play in the West division, the *WDW* will have rank 11. In 1994, when the season ended early because of a players' strike, the Texas Rangers were the first-place team in the West and were also ranked 11th out of the 14 teams on the basis of winning percentage. So this worst-case scenario is not that farfetched.

The expected rank of the worst divisional winner is given by

$$E\left(WDW\right)=\sum_{n=3}^{11} n \ \ P\left(WDW=n\right).$$

The probabilities in this sum are once again found by making conditional which division contains team n. Specifically,

$$\begin{aligned} P(WDW=n) &= P(WDW=n \text{ and team } n \text{ wins East}) \\ &+ P(WDW=n \text{ and team } n \text{ wins Central}) \\ &+ P(WDW=n \text{ and team } n \text{ wins West}). \end{aligned}$$

For ease of writing, let $CW(n)$ represent the event that the teams ranked from 1 to $n-1$ are all in either the Central or West division, with both divisions having at least one of these teams. Then, for example, we have

$P(WDW=n$ and team n wins East$)=P($team n wins East given that $CW(n)$ occurs$)\cdot P(CW(n)$ occurs$)$.

Table 1 Distribution of Ranking of Worst Divisional Winner (*WDW*)

n	$P(WDW = n)$
3	.2747
4	.2747
5	.1998
6	.1249
7	.0699
8	.0350
9	.0150
10	.0050
11	.0010

To apply this formula, consider the case $n = 5$. If we know that teams 1–4 are in the Central and West divisions, then there are 10 teams remaining, and 5 of them are in the East division. So we see that

$$\begin{aligned} &P(\text{team 5 wins East given that } CW(5) \text{ occurs}) \\ &\qquad = 5/10 = 1/2. \end{aligned}$$

We now calculate the probability that $C(n)$ occurs. The total number of ways to place teams 1–4 across all the divisions is $14 \cdot 13 \cdot 12 \cdot 11$. Now, there are $9 \cdot 8 \cdot 7 \cdot 6$ ways to place teams 1–4 in only the Central or West division. Of these, there are $5 \cdot 4 \cdot 3 \cdot 2$ ways in which all 4 teams are placed in the Central division, and $4 \cdot 3 \cdot 2 \cdot 1$ ways in which all the teams are in the West division. Because $C(n)$ requires that both divisions have at least one team, these last two cases must be excluded. Hence, we find that

$$\begin{aligned} &P\left(CW(5) \text{ occurs}\right) \\ &\qquad =\frac{9\ 8\ 7\ 6-5\ 4\ 3\ 2-4\ 3\ 2\ 1}{14\ 13\ 12\ 11}=\frac{2880}{24024}. \end{aligned}$$

Putting these two probabilities together, we see that

$$\begin{aligned} &P(WDW=n \text{ and team } n \text{ wins East}) \\ &\qquad = (1/2)(2880/24024) \approx .05994. \end{aligned}$$

With similar work, we can find that

$$P(WDW=5) = .05994 + .05994 + .07992 \approx .1998.$$

Using the same method, we can calculate the values of $P(WDW=n)$ in **table 1**. Note that the sum of the last four probabilities in **table 1** is .056. Hence, the worst divisional winner will fall in the bottom half of the league 5.6% of the time. Using the formula for the expected value, we have

$$E\left(WDW\right)=\sum_{n=3}^{11} n \ \ P\left(WDW=n\right)\approx 4.6364.$$

Thus, by the linearity of the E operator, $E(WDW - WC) = E(WDW) - E(WC) \approx 1.6474$. We expect that, on average, the wild-card team will rank 1.6474 spots ahead of the worst division winner in the overall league standings.

COMPARING THE MODEL TO REALITY

By reviewing the standings of the American League from 1994 through 2011, we can compare the average results of the probabilistic model to what actually occurred during those 18 seasons. These comparisons are shown in **table 2**.

Table 2 Model Estimates and Actual American League Values

	$E(WC)$	$E(WDW)$	$P(WC < WDW)$
Model estimate	2.989	4.636	.725
Actual outcome	2.917	4.361	.722

THE NATIONAL LEAGUE AND RECENT CHANGES

The techniques we have used here can, of course, be applied to the National League as well. Unlike the American League, the National League has 16 teams, with 5 each in the East and West divisions and 6 in the Central division.

Here are the results:

$$E(WC) \approx 2.976,\ E(WDW) \approx 4.716, \text{ and } P(WC < WDW) \approx .732$$

Also, the worst divisional winner should fall in the bottom half of the National League about 2.9% of the time. In reality, from 1994 through 2011, the average rank of the wild-card team in the National League was 3.139, and the average rank of the worst divisional winner was 4.389. In addition, the wild-card team was better than the worst divisional winner, with probability .556.

In 2013, the Houston Astros moved to the American League West, so both major leagues now have three divisions of 5 teams each. Moreover, beginning in 2012, each league now sends two wild-card teams to the playoffs. You may wish to have your students study the analyses presented here and challenge them to do a similar analysis for the current divisional and playoff structure for the MLB.

SHORTCOMINGS OF THE MODEL

Our proposed model is distinctive in its simplicity. Its primary assumption is that teams in a league are to be rank-ordered and then randomly assigned to three divisions, whether they are the better teams, lesser teams, or middling teams. With this one main assumption, the model is able to produce important numerical outcomes that are consistent with actual values stemming from the MLB seasons from 1994 through 2011. It does this without the aid of any player statistics or team statistics. But the model's purpose is limited—comparing the strength of the weakest divisional winner with that of the wild-card team.

Statisticians George Box and Norman Draper once wrote, "All models are wrong, but some are useful" (Box and Draper 1987, p. 424). And, indeed, our proposed model is not perfect. For instance, in real life, the team rankings do not appear to be randomly distributed throughout the divisions. In the American League, for example, consider the top two teams from each of the 18 seasons. Of those 36 teams, 20 have been in the East division. Randomness implies that approximately 13 teams should have been from the East. Much of this discrepancy is due to the combined impact of the dominance of the Yankees and the Red Sox franchises. Yet the model still performs admirably when comparing the wild-card team with the worst divisional winner. This result may be due to the robustness of the model over such discrepancies, or it may be due to plain luck.

CONCLUSIONS AND FURTHER EXPLORATION

Although the objections noted by many baseball enthusiasts are understandable, these results of our model suggest that the worst divisional winner—not the wild-card team—is typically the weakest link in the three-division playoffs structure. This finding seems to contradict Johnny Bench's belief that wild-card playoffs elevate mediocrity. Rather, it seems that the average quality of the playoff teams increases when the wild-card team is included.

Readers interested in learning more about using probability models in baseball applications are encouraged to read two excellent books: *Teaching Statistics Using Baseball* (Albert 2003) and *A Mathematician at the Ballpark* (Ross 2007). Teachers may have their students apply the methods of this article to verify the National League results. Or students may even apply these methods to Major League Baseball's new system of having two wild-card teams in each league.

REFERENCES

Albert, Jim. 2003. *Teaching Statistics Using Baseball.* Washington, DC: Mathematical Association of America.

Box, George E. P., and Norman R. Draper. 1987. *Empirical Model Building and Response Surfaces.* New York: John Wiley and Sons.

Hummel, Rick. 2006. "Around the Horn." *St. Louis Post-Dispatch*, June 4, sec. C, p. 6.

Lawes, Rick. 1992. "Playoff Expansion: A New View." *USA Today's Baseball Weekly* 30 (2): 3.

Ross, Ken. 2007. *A Mathematician at the Ballpark: Odds and Probabilities for Baseball Fans.* New York: Plume, The Penguin Group.

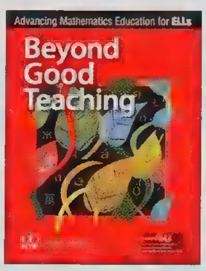
RICHARD E. AUER, rea@loyola.edu, and **MICHAEL P. KNAPP,** mpknapp@loyola.edu, are colleagues at Loyola University Maryland. Auer enjoys working with students on summer research projects, usually on statistical modeling of baseball and other sports. Knapp is a number theorist who dabbles in probability.

FOR YOUR information

PUBLICATIONS

From NCTM
Individual NCTM members receive a 20 percent discount on NCTM publications. To order, visit the NCTM online catalog at www.nctm.org/catalog or call toll free at (800) 235-7566. Free print catalogs of NCTM publications also are available by writing to NCTM.

Beyond Good Teaching: Advancing Mathematics Education for ELLs, Sylvia Celedon-Pattichis and Nora G. Ramirez, 2012. 236 pp, $35.95 paper. Grades pre-K–2, 3–5, 6–8, 9–12. ISBN 978-0-87353-688-2. Stock no. 14118. National Council of Teachers of Mathematics; www.nctm.org.



This collection of personal reflections, vignettes, case studies, and research provides a comprehensive view of the struggles that ELL students and teachers face in the classroom every day. The book offers some excellent suggestions for ways in which ELLs' inclusion and success can be accomplished while maintaining a rigorous, standards-based mathematics curriculum.

Beyond Good Teaching begins with personal reflections of English language learners and their teachers sharing their experiences in ELL education. Questions in the margins guide readers in reflecting on their own experiences and practices. A downloadable page is provided online to help with organizing and recording reactions.

In subsequent chapters, the stories from students, teachers, counselors, administrators, and parents create a framework for understanding the dynamics of a classroom that contains ELLs. Chapters focus on various barriers and challenges to educating ELLs and offer research- and experience-based strategies for assisting them in the classroom. Topics include assessment, language development, maintaining standards, and analysis of effective lessons.

As an educator in a region where ELL students in the classroom is a norm, I highly recommend this book as a professional development tool for educators at all levels of experience and for all grade levels. It offers great insight into issues that can arise when teaching mathematics to ELLs.

—Catherine Tabor
Coronado High School
El Paso, TX

Prices of software, books, and materials are subject to change. Consult the suppliers for the current prices. The comments reflect the reviewers' opinions and do not imply endorsement by the National Council of Teachers of Mathematics.

Teaching Mathematics for Social Justice: Conversations with Educators, Anita A. Wager and David W. Stinson, eds., 2012. 213 pp., $36.95 paper. ISBN 978-0-87353-679-0. Stock no. 13955. National Council of Teachers of Mathematics; www.nctm.org.

How can mathematics education be used to challenge injustice in society? This excellent resource for social justice mathematics presents some of the history and philosophy of the movement along with ideas for incorporating mathematics for social justice into courses for student and teacher education.

Social justice mathematics cannot be described simply. Teaching mathematics for social justice encompasses everything from teaching disadvantaged students how to reason mathematically to using mathematics to analyze specific issues of social injustice. The book's history and philosophy section is informative but, unfortunately, lacks focus.

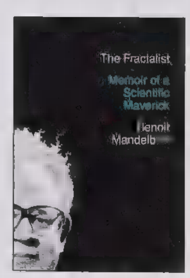
The strongest parts of the book are the chapters describing classroom experiences. These four chapters present some interesting specific topics while also discussing the challenges of finding subjects that will interest students, making time for the material, managing discussion of controversial topics, and dealing with skeptical supervisors. Topics include the cost of wars, inequality in societies, college loans, foreclosures, election results, racial profiling, and domestic violence.

Every mathematics course requires numerous application problems that can include social justice topics. Arithmetic, statistics, and the mathematics of finance, for example, afford many opportunities to incorporate social justice mathematics. Projects are another channel for addressing such topics. Other courses, however, present more of a challenge.

—Thomas Sonnabend
Montgomery College
Rockville, MD

FROM OTHER PUBLISHERS

The Fractalist: Memoir of a Scientific Maverick, Benoit Mandelbrot, 2012. 324 pp., $30.00 cloth. ISBN 978-0-307-37735-7. Pantheon Books; www.randomhouse.com.



"Bottomless wonders spring from simple rules … repeated without end."—Benoit Mandelbrot, from his last major talk, 2010 Technology, Entertainment, and Design (TED) conference.

Mandelbrot, perhaps the greatest mathematician of our times, is well known among geometry teachers and students as the father of fractal geometry. What is not as well known is the story behind the mathematician. *The Fractalist* is Mandelbrot's memoir, written shortly before he died in 2010 at age 85, having never had the opportunity to make final revisions.

Born in 1924 in Warsaw, Mandelbrot fled with his family to Paris in 1936, just

ahead of Hitler's occupation. His story of his Jewish family's survival in France during World War II is fascinating, and his need to keep moving at that time perhaps explains his constant movement during his professional career.

Mandelbrot's education and early career choices were heavily influenced by his uncle Szolem, also a prominent mathematician. Szolem pointed him in the direction of his initial research and his first "Kepler moment," which he describes as "bringing to the field the element of rational mathematical structure that Kepler brought to physics several centuries before."

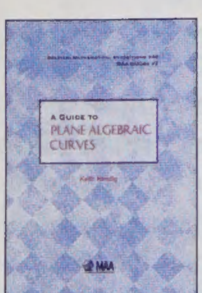
After education in Paris, Mandelbrot attended Caltech, MIT, and Princeton University. He landed at IBM Research, where he worked for thirty-five years. During this time, he took leaves to teach economics, engineering, mathematics, and physics at Harvard, MIT, and Yale. The father of fractal geometry could easily have been as well known in a number of other fields.

Mandelbrot's life centered on his professional work, and his memoir reflects that emphasis; it includes very little about his personal life, interests, and family and remarkably few anecdotes about his professional colleagues. Further, the book contains surprisingly little mathematics; many readers would have welcomed more.

Reading Mandelbrot's memoir made me want to learn more about his work, his life, and the mathematics that he discovered.

—David Ebert
Oregon High School
Oregon, WI

A Guide to Plane Algebraic Curves, Keith Kendig, 2011. 208 pp., $49.95 cloth. ISBN 978-0-88385-353-5. Mathematical Association of America; www.maa.org.



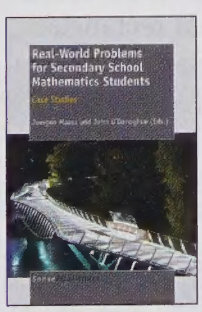
The advent of powerful computer software in the past twenty years has led to a revival in the study of algebraic curves and algebraic geometry, the applications of which can be found in fields such as coding theory, dynamical systems, and robotics. This book is a straightforward and simple introduction to plane algebraic curves and would be of interest to anyone wanting a good overview of the subject. The material could also serve as a useful supplement for students taking an introductory course on algebraic curves or for mathematicians who would like to learn about the subject. Even though the book is written as a guide, readers will need some basic understanding of complex analysis, field theory, and topology to comprehend the subject matter completely.

The book is informal in its approach and focuses on building intuition through a variety of pictures and clarifying examples without becoming mired in detailed proofs and exercises. Chapter 1 explores algebraic curves in the real plane; in subsequent chapters, the canvas expands to include the complex numbers. The book's final chapters focus more on the geometric properties of algebraic curves and conclude with a foray into the topic of Riemann surfaces.

A Guide to Plane Algebraic Curves is an accessible and well-written book that anyone with an interest in this beautiful subject will surely appreciate and find useful.

—Marc Michael
Frostburg State University
Frostburg, MD

Real-World Problems for Secondary School Mathematics Students: Case Studies, Juergen Maasz and John O'Donoghue, eds., 2011. 292 pp., $49.00 paper. ISBN 978-94-6091-541-3. SensePublishers; www.SensePublishers.com.



This book, intended for secondary school teachers and their students, presents fourteen in-depth descriptions and analyses of real-world situations that could be used in secondary school classrooms. Two additional chapters focus on the general notion of incorporating everyday situations into the mathematics classroom and on using technology as a tool in the modeling process.

Topics range from digital imagery, space travel, and production processes to coding theory, polls and surveys, and the mathematics of eggs! A significant strength of most chapters is the extensive background, detail, and analysis provided for the topic. The nineteen-page chapter on coding theory, for instance, includes extensive mathematical discussion of bar codes, communication of digital images from space, and compact discs. In chapter 8, using NBA basketball to motivate modeling, the authors provide a multifaceted analysis of shot trajectory, devoting more than half the chapter to this topic.

Although many authors mention classroom uses for the topics and acknowledge the likely need for teachers to reflect carefully on how to shape their classrooms for these sorts of investigations, readers may be frustrated to find relatively little direct attention to classroom implementation. In the coding theory chapter, for instance, the "To Do in Class" section is only two paragraphs long; it gives a description of a project to consider but provides no materials or examples. This scant treatment is not uncommon throughout the book.

If you are searching for a collection of real-world mathematics topics and are motivated to reinforce and extend your mathematics knowledge through careful study, this book is for you. The next step—classroom incorporation and implementation—will be your responsibility, because the authors provide few specific suggestions and examples.

—Roger Day
Illinois State University
Normal, IL

GIVE US YOUR SUGGESTIONS

If you know of a new book or product that you wish to see reviewed in *MT*, let us know. Send information about the item, including its title and the publisher's name, to **mtreviews@nctm.org**. We will be sure to consider the item for FYI.

MY FAVORITE lesson

Judith Macks

The Dog Pen Problem

My favorite lesson is the Dog Pen problem:

> Suppose that you had 64 meters of fencing with which to build a rectangular pen for your large dog.
>
> - What are the dimensions of some different pens that you can make if you use all the fencing?
> - What dimensions are best to allow the most space for the dog to run?
> - What dimensions will allow the most play area? What dimensions allow the least play area?

This problem is quite flexible, with variations that can address different learning objectives. Further, the discussions before, during, and after work on the problem can focus on almost every one of the Common Core Standards for Mathematical Practice. The problem requires little background knowledge, is based on a familiar situation, and has many entry points.

Although the problem asks only for "some different pens," questioning during work time usually results in lists of at least all whole-number possibilities (see **fig. 1**). Students generally arrive at solutions by using some combination of strategies: drawing a picture, creating a table, guessing and testing, following a pattern, or using variables. Discussion

Width	Length	Area
1 m	31 m	31 m^2
2 m	30 m	60 m^2
3 m	29 m	87 m^2
4 m	28 m	112 m^2
5 m	27 m	135 m^2
⋮	⋮	⋮
14 m	18 m	252 m^2
15 m	17 m	255 m^2
16 m	16 m	256 m^2
17 m	15 m	255 m^2
18 m	14 m	252 m^2
⋮	⋮	⋮

Fig. 1 This list includes some of the whole-number solutions to the Dog Pen problem.

during and after the student work session encourages good use of mathematics vocabulary, which helps improve student understanding of area and perimeter. Further, by the end of the discussion, students know that rectangles with the same perimeter may not be congruent or have the same area; thus, their understanding of how perimeter, area, and congruence relate to each other has increased. They also see that there can be more than one way to solve a problem successfully and that problems can have more than one accurate answer. But the learning does not have to stop there.

Questions about patterns can lead to rich discussions that help develop students' expertise in looking for structure. Although students may not readily notice any pattern in the sequence of areas of pens with integral sides, questioning usually helps some notice that differences in areas form a sequence of consecutive odd numbers.

To explore still more variations on this lesson, consider adding questions such as these:

- Which rectangular pen has the largest area? Does your result hold for any length of fence?
- What about pens that are not rectangles? Do any of them have a larger area?
- If x is either the length or the width and $f(x)$ is the area, what does the graph of $f(x)$ look like?

I am still discovering new questions to ask about this problem. After reading Steven Siegel's "The Ratio of Perimeter to Area" (Reader Reflections, *MT* May 2010, vol. 103, no. 9, p. 632), I look forward to asking students to use the Dog Pen problem to investigate Siegel's concept of *density*: "the ratio of perimeter to the area of the region it bounds."

I have also used a variation of the lesson to focus on pedagogy when working with preservice and in-service teachers. Asking teachers first to find the area and perimeter of a square and a nonsquare rectangle and then to compare these tasks with the Dog Pen problem demonstrates clearly the difference between an exercise and a true problem.

The Back Page provides a forum for readers to share a favorite lesson. Lessons to be considered for publication should be submitted to mt.msubmit.net. Lessons should not exceed 600 words and are subject to abridgment.

Edited by ***Jennifer Wexler,*** *wexlerj@newtrier.k12.il.us, New Trier High School, Winnetka, IL*

JUDITH MACKS, jmacks@towson.edu, is a lecturer in the mathematics department at Towson University in Maryland. She is interested in improving the learning and teaching of mathematics by working with preservice and in-service mathematics teachers.